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E U C L I D's
ELEMENTS

O F
G E O M E T R Y,
The First Six,
the Eleventh and Twelfth Books;

Translated into ENGLISH,
From Dr. GREGORY's EDITION,
WITH
NOTES and ADDITIONS.

For the Use of the BRITISH YOUTH.

By E. STONE.

Ptolemy, King of Egypt having asked Euclid whether there was any other more compendious way of arriving at Geometry, than by these his Elements, is said to have answered, Μη εἶναι βασιλικὴν ἀτραπὸν πρὸς γεωμετρίαν, There is no other way or royal passage to Geometry.

Proclus's Commentary upon Euclid's second book.

L O N D O N,

Printed for and sold by THO. PAYNE, next the *Mews Gate*, in *Castle-street*, near St. Martin's Church;

Sold also by Mr. JAMES FLETCHER at *Oxford*, Mr. THURLBOURN at *Cambridge*, Mr. HILDYARD at *York*, Mr. NEWTON at *Manchester*, and by Mess. HAMILTON and BALFOUR at *Edinburgh*.

MDCCLII.

EXTRA
SUPPLEMENT

OF THE

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TO HIS
ROYAL HIGHNESS
THE
Duke of Cumberland.

S I R,

WHENEVER I considered
the small importance of my-
self, the greatness of your rank, and
the little value of my share in the
following work, I was always de-
terr'd from attempting to offer it to

A 2

Your

Your favour and protection. But when I reflected that the ancient, celebrated, and, I may say, noble and useful Elements of Geometry of Euclid alone, are the principal source and main spring of all mathematical learning, and whatever valuable knowledge is thence derived; that they have always been referred to by the greatest masters of the mathematics, and held in the greatest esteem for these two thousand years, by all ranks and degrees of men. — When I observed with regret, that the avenues, passages, and paths of the kingdoms of profound learning and useful literature; so full of tall cedars, sturdy oaks, beautiful plants, rich fruits, and pleasant flowers, have been, for these many years last past, so filled, clogged up, overgrown, and interrupted with the disagreeable bushes, prickly

DEDICATION. v

prickly briars, unprofitable and poisonous weeds, of Jacobite and republican politics; the printed twelve-penny productions of crafty knaves, and angry gentlemen, discourses of wrong-headed parsons, and foolish tradesmen, the two penny letters in the weekly news-papers, etc. all highly and manifestly prejudicing and lessening true learning, disgracing the kingdom, propagating error, and promoting discord and confusion amongst us. When I considered that all the foreign nations of figure in Europe at this day have their public schools and academies for learning their military officers, nobles, and gentry, the mathematics, well knowing their great benefits and advantages — When I considered what numbers there are of brave young men, our British officers, and other gentlemen, all of sufficient

A 3 spirit,

vi DEDICATION.

spirit, capacity, and leisure to acquire any knowledge whatsoever, and how necessary geometry and mechanics are to the employments of the former; and tho' perhaps not such in so great a degree to the latter, yet at least are rational and useful amusements, that would innocently employ their time, and somewhat take them off from irksome idleness, vice, and the too great enjoyment of sensual pleasures, hurtful to their bodies, minds, and estates.——
Lastly, when I considered that one kind of knowledge or learning is cultivated and pursued by greater numbers, or is more in fashion at one time than another; that (as I said before) the study of politics has been too long predominant, and too much in fashion amongst all ranks of people; and that it is in the power of princes, and other grandees of a kingdom, to
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DEDICATION. vii

alter the fashion of learning as well as that of cloaths, by their influence, example, and imitation. — I say, when I consider'd all this, I was encouraged to encounter and break thro' my resolution first mentioned, and accordingly have presumed to offer these Eight books of Euclid's Elements of Geometry, as to one by whose influence, power, command, and desire, all variously operating, whatever of mathematics and other useful learning during this age of unprofitable tho' fashionable politics, has been lost, or at a stand, may thro' You be restor'd, rendered more flourishing than ever, and again become the favourable and fashionable knowledge amongst us. If to this likewise I shall in the least contribute, by what I here offer, who in reality as a poor labourer (only destin'd to remove dirt and rubbish) imperfectly,

A 4

viii DEDICATION.

fectly, unskilfully, and presumptuously, points out to a great architect, the beauties of a spacious palace, and the injuries it may have received, and is now exposed to, I shall be as highly pleased, as I am proud of subscribing myself,

SIR,

Your ROYAL HIGHNESS's

Most Devoted,

Most Humble,

and Obedient Servant,

EDMUND STONE.

PREFACE.

WISDOM is certainly one of the main springs of human happiness. And the more of it each individual of a society acquires, the more happy, all things else alike, will that society be; for the wiser a man is, the better will he know how to preserve his own health, and pleasantly lengthen out his own life: he will better avoid the injuries of weather: better provide food and clothing: better avoid pain and sickness: better soothe, lessen, and command his own and other people's passions: better observe the laws of his country; and in short better promote his own and neighbours happiness in all respects forever. Accordingly since wisdom is so necessary to happiness, that learning, best adapted to promote and extend it, is undoubtedly the most valuable, and ought by all to be most encouraged, cultivated, and used. And therefore, since true wisdom very much consists in, and is obtained from, the knowledge of the comparative numbers and magnitudes of the qualities, powers, efficacies, matter, and motions of sensible objects, their various increase and decrease, &c. and as these may be acquired in a great measure from arithmetic and geometry: these two sciences may truly be said to be the two great fountains from whence much of the wisdom and happiness of mankind flow: And therefore, if possible, they should be learned by every member of a society. No state ever did or scarcely could flourish without them. They have always been in esteem and cultivated proportionably to the greatness and riches of a country, and as these have declined so have those.

When

When learning flourish'd amongst the Romans, in the time of Cicero, mathematics was honoured and esteemed, and that by this great man himself; but afterwards, in worse times, viz. the reign of Nero, &c. when gross ignorance had seized the state, and learning became very weak and decrepid, the mathematics was neither understood nor in any repute, and some of the degenerate writers of those times, such as *Tacitus*, *Suetonius*, *Gellius*, seem'd even not to know what mathematics or mathematicians were; they ranked mathematicians with conjurers and fortune-tellers. Nay it is said in *Tacitus's Annals*, that the senate pass'd a decree for banishing the mathematicians out of Italy.

As tyrants and arbitrary governours, immerf'd in pride, and stain'd with madness, are generally too stupid and brutal to be capable of understanding and esteeming these sciences, and their excellencies, and the major part of crafty Jesuits and other spiritual jugglers, retailers of nonsense and spreaders of lies, have been always aspersing and lessening them, because by their evidence, certainty, and use, they are so apt to dispose the mind to discover truth, and so much straiten, and clog up the paths and passages of fraud and implicate faith; so on the other hand, all rational rulers, friends to liberty, and lovers of wisdom and mankind, detesters of falshood, and despisers of fraud, have ever imbraced, and always encouraged these sciences; well knowing them to be the plentiful fountains of advantage to human affairs, from whence, in a great measure, spring the principal delights of life, securities of health, increase of fortune, and convenience of labour; which habituate the mind to a constant delight in study; which strengthen and subject it to the government of right reason; which wonderfully fortify it against all kind of imposition; and make it more easily, readily, and

and powerfully encounter and overcome falshood, wherever it appears.

I have often thought, that the time of too many of the youth of this kingdom has long been, and now is misapplied, in learning Latin and Greek, especially the latter, and believe it would be of much more general advantage for greater numbers of them to be more employed in learning arithmetic, geometry, mechanics, and natural philosophy; as also rational systems of morality, politics, and good government: certainly some or all of these should be known, in a competent degree, more especially by all statesmen, lawyers, soldiers, sailors, physicians, surgeons, artificers, &c. whereby they would all be better fitted for their several employments, know more of them, and execute them with much more ease, exactness and certainty. So that as *Plato*, the philosopher, wrote over the the door of his academy, *Let none enter in here but those that understand Geometry*, I would have it wrote over the door of every academy and public school in Britain,

Let none depart from hence unskilled in Mathematical science.

It would require a volume to distinctly discuss, and fully lay out the beauties, excellencies, and uses of these sciences, as well as an abler hand than mine to do it; however if any body is suspicious of the truth of these naked assertions or general positions, and should call upon me for a particular proof, I shall at all times be both ready and willing to do it. Being firmly persuaded, that the truth of every one of them can be proved by such arguments, as cannot but be convincing to every rational man.

As use has a most powerful influence over the mind, as well as the body, there is one certain advantage derived from the study of geometry especially, if it were no more, for which it always ought
to

to be highly valued and held in the greatest esteem, viz. by constantly searching after and demonstrating geometrical truths, the mind of a geometrician is so much employ'd in, and used to truth, that it becomes, as it were, a part of his geometrical constitution: he ever loves it, and is constantly seeking it, in all sorts of subjects as well as those of geometry: he is so conversant in truth, that he naturally hates falshood, tho' perhaps he sometimes is obliged, contrary to his inclination, to make use of it—hence, since it is demonstrable that the happiness and preservation of a society is augmented by the greater quantity of veracity dispersed throughout the individuals of that society, for lying and deceit tend to destroy it; who does not see the great use of the study of geometry in the promotion and maintenance of truth, and consequently its tendency to the preservation of society itself. Even the * author of a discourse, called *The Analyst*, printed in the year 1734. altho' he is so full of mistakes, and so much lessens and disparages fluxions and the mathematical students therein, and that chiefly because he does not understand them, has done justice to geometry, and spoken truly thereof. He says, “Geometry is an excellent logic, and it must be own'd, that when the definitions are clear; when the postulata cannot be refused, nor the axioms denied; when from the distinct contemplation and comparison of figures, their properties are derived by a perpetual well-connected chain of consequences, the objects being still kept in view, and the attention ever fixed upon them, there is acquired an habit of reasoning, close, and exact, and methodical: which habit strengthens and sharpens the mind, and being transfered to other subjects is of general use in the inquiry after truth.”

* The present Bishop of Cloyne, as I have heard.

But

But enough of this in general; let us come nearer our present work, and more particularly observe, that we Britons have had amongst us scarcely any who have ventured to write Elements of Geometry of their own; altho' this nation has produced both the greatest mathematicians, and the greatest number of them too. We have generally liked Euclid's Elements best, and know them too well to be at the trouble of compiling new Elements of Geometry (as many of the French, and other foreigners have done) and thereby changing better for worse. We think it far more eligible to do nothing at all, than vainly busy ourselves in matters introducing confusion and error into the geometrical world, and by avoiding false praise, have likewise been free from evident disgrace, and real dishonour. For the method and order of Euclid's Elements of Geometry, can neither be mended nor altered, but for the worse. And the demonstrations are mostly so very accurate, nervous, and elegant, as not to be equalled by any geometrical writer whatsoever, either ancient or modern. His method may be defended for ever, and his demonstrations will be approved by all men of sound judgment to the end of the world. His first principles or axioms are few, simple, and clear, taken from our primitive and natural conceptions of things, and such as every one can easily apprehend, and no man in his senses can deny. And his method is such, that nothing is taken as true, unless it be demonstrated, and nothing is demonstrated, but from what went before. And the demonstrations, as I said before, are in the main so perfect and compleat, that the most severe critic could never find a real fault in them. The greatest masters of reasoning have always been captivated and charmed by their beauty and elegance, and, as one may say, they ravish the reader's assent, and force an absolute command

mand over the mind that dares encounter them.

But other element writers, whether such as have alter'd Euclid's order; such as have given other demonstrations of some of his propositions; such as have left out some of the most simple propositions, or ranked them amongst the class of axioms, are generally faulty in some respect or other, their own demonstrations are oftentimes imperfect, or sophistical, and sometimes obscured and vitiated by the mixture of algebra and algebraical signs. In a word, the Elements of Euclid far exceed them all, especially as to method and elegance of demonstration.

As to the fifteen books of Euclid's Elements, it is true there are some of more importance and use than others in the geometry requisite to the necessary mechanic arts, and useful reputable sciences, now exercised and cultivated amongst the several nations of Europe. The seventh, eighth, and ninth books are pretty short elements of arithmetic, and not of geometry; altho' these, with the rest, are altogether usually called Euclid's Elements of Geometry. The tenth book, being the Elements of the doctrine of incommensurability, is indeed fine, but its length, and apparent inutility in any of the favourite mathematical studies of these ages makes it very unpalatable, and much neglected. The same may almost be said of the 13th, 14th, and 15th books, containing the theory of the five regular solids, or Platonic bodies, as they are called. What divinity the antients found in these bodies I cannot at all imagine; surely there must have been something very extraordinary in them, for Euclid, is expressly related by *Proclus* to have compiled the whole system of his Elements only for the sake of the doctrine of the five regular solids. However it must be owned, that these books, tho' elegant in themselves, and it may be presum'd sufficiently

sufficiently valuable for the purposes they were intended by Euclid himself, do not so nearly belong to the elements of plane and solid geometry, as the first six, eleventh, and twelfth books.

Accordingly these eight books alone by most of the moderns have been looked upon as sufficient Elements of that plane and solid geometry, in use and fashion amongst us.

If it should be asked, what occasion there was for publishing afresh another English translation of these eight books, when there are so many already, especially after the several English impressions of *Commandine's Euclid*; I answer, this translation is from the best and most correct edition of Euclid himself by Dr. Gregory, who himself says, in the preface, that his own Latin translation is vastly more correct than *Commandine's*; that Dr. Hudson carefully compared the copy with the best original Greek manuscripts, given by Sir Henry Saville for the use of the university of Oxford; and diligently revised the sheets over and over before they were printed off, as they came from the press.

The figures are annexed to the several propositions, and not at a distance. Those of the fifth book are more distinct and better adapted to the generality of the propositions, and oftentimes better suited to the comprehension of the demonstrations. The notes which I have added do explain and clear up some difficulties and obscurities that may occur to learners, and easier demonstrate some propositions, and clear Euclid from some seeming faults and oversights; for I am not entirely of opinion with Dr. Keil, who says, in his preface to *Commandine's Euclid*, that Euclid himself is clearer than any of his commentators. Indeed tho' *Clavius's* Commentary upon *Euclid*, as the Dr. rightly observes, is in general certainly too tedious and prolix, yet many of his observations are useful, and make *Euclid* the

the easier. I never in my life knew a learner that did understand *Euclid's* fifth definition of the fifth book, without a further explanation.

I have also added several propositions to this edition, containing many valuable, useful, and elegant theorems and problems, which, with those of *Euclid* himself, do render the whole more compleat elements of common geometry; and some of these are new, at least to me; such as Prop. ii. at page 54. Prop. vii. at page 64. the Scholium at page 97. the Scholium at page 100. Prop. viii. at page 204. Prop. ix. at page 205. the Scholium at page 106. and the proposition. Prop. ix. at page 158. Prop. x. at page 160. Prop. xi. at page 161. Prop. xii. at page 163. Prop. xiii. at page 164. Prop. vi. page 197. the theorem at the end of page 273. Prop. viii. page 298. and several others.

I hope the errors of the press are but few. An honest able friend of mine*, very skilful in geometry, having assisted me in correcting the sheets as they came from the press. The demonstrations of the lemma at page 102. of Prop. xxiii. of the sixth book at page 273. and of Prop. ii, of the additions to the sixth book, at page 291, are his.

* Mr. *William Payne*, a teacher of mathematics.

E R R A T A.

Page 99. line 25.	for A D read A B.
119.	31. dele equal.
132.	2. for G B D read G D E.
178.	10. for B C read B G.
179.	33. for 4. 1. read 1. 4.
205.	10. read submultiple.
207.	36. for of either read either of.
218.	7. read equimultiple.
372.	12. for Cavillarius read Cavallerius.

EUCLID's

E U C L I D ' s E L E M E N T S. B O O K I.

D E F I N I T I O N S.

1. **A** POINT is that which has no parts ^a.
2. A line is length without breadth ^b.
3. The extremes of a line are points.
4. A right line is that which lies evenly between its points ^c.
5. A superficies is that which has only length and breadth ^d.
6. The bounds of a superficies are lines ^e.
7. A plain superficies is that which lies evenly between its lines ^f.

^a It may perhaps be as well to say, A point is that which is less than any assignable or even conceiveable magnitude.

^b A line may be conceived to be generated or produced by the motion of a point.

^c Some call a right line the shortest of all lines that have the same extreme points or bounds; others, that it is that line whose parts all tend the same way, or lie in the same direction, or of which no point in it is raised or depressed.

^d Do not those describe a superficies best, who say it is what is generated by the motion of a line, not moving endways or in the direction of itself?

^e This, as well as the third definition, of the extremes of a line, are not so much definitions, as necessary consequences and common notions, arising from the definitions of a line and a superficies, there being no new geometrical term defined in either of them.

^f Some call a plain superficies, that which is the least of all

8. A plain angle is the mutual inclination of two lines to one another in the same plane, so touching each other as not both to lie in the same right line ^g.

9. If the Lines containing an angle be right lines, that angle is called a right lined angle ^h.

10. When a right line standing upon a right line makes angles on each side thereof equal to one another, each of these equal angles is a right angle; and that right line which stands upon the other is called a perpendicular to that upon which it stands.

11. An obtuse angle is that which is greater than a right angle.

12. An acute angle is that which is less than a right angle.

13. A term is that which is the extreme of any thing ⁱ.

14. A figure is that which is contained under one or more terms.

those having the same bounds or bound; or that which is generated by the motion of a right line not moving according to the direction of itself.—Others, that to all of whose parts a right line agrees; or, that whereof there is no part raised or depressed.

^g I am inclined to think that Euclid himself, in this definition, only meant a right-lined plain angle, and not all sorts of plain angles; and that the word right-lined was left out by the carelessness of transcribers; for angles of contact, whether of a curve line and a right line, or of two curve lines, are not in reality comprehended under the words of this definition; since the lines forming these angles at the point of contact, do both lie in the same direction; when, at the same time, the definition expressly says they should not. The notion of a plain angle, perhaps, may as well be conveyed, by saying, it is the aperture or opening of two lines meeting one another in the same plane; but not so as themselves or their continuations to coincide or fall into each other, that is, both become one line.

^h Some say, a straight or right-lined angle is a portion of a plain space about a point contained by two right lines, drawn from the same point, and not directly situated to one another.

ⁱ This is not properly a definition, or at least it is needless; since, if it were omitted, nothing that follows would be either imperfect or obscure; and it is thought not to be Euclid's originally, but rather a by-note of some lexicographer, than the definition of a geometrician.

15. A circle is a plain figure contained under one line, which is called the circumference; to which all right lines drawn from a certain point within the figure are equal ^k.

16. And that point is called the centre of the circle.

17. A diameter of a circle is a right line drawn through the centre, and terminated on each side by the circumference, and divides the circle into two equal parts ^l.

18. A semicircle is a figure contained under a diameter, and that part of the circumference of a circle cut off by that diameter ^m.

19. A segment of a circle is a figure contained under a right line, and part of the circumference of a circle ⁿ.

20. Right-lined figures are those which are comprehended by right lines.

21. Three-sided figures are such as are contained by three right lines ^o.

22. Four-sided figures are such as are contained by four right lines.

23. Many-sided figures are such as are comprehended by more right lines than four.

^k Some would rather have a circle defined from its generation, whereby both its existence and the nature of the figure may be declared together; as thus, A circle is a figure described in a plane by the revolution of a right line about one of its extreme points remaining fixed, till the right line returns to the place from whence it first began to move.—Dr. Barrow, and other able geometers, say in general, that definitions of things from their generations are always best and most natural, if they can be given with any elegance, very much conducing to the discoveries of their essential properties.

^l The latter part of this definition, that the diameter cuts the circle into halves, or two equal parts, is both unnecessary, and a demonstrable proposition; for which reason, some will not have it to be Euclid's, but some marginal note, how old soever, that happened to be transcribed into the text.

^m Is not this definition of a semicircle needless? for, in reality, it only signifies half a circle. By a like reason, a semidiameter, or half diameter, might as well have been defined,

ⁿ This definition of the segment of a circle is misplaced, it being not used in the first and second books of Euclid. It is again repeated at def. 5. of the third book, where it is in its proper place.

^o The side of a figure should be defined before it is used; it is one of the lines that comprehend the figure.

24. Of three-sided figures, that is an equilateral triangle, which has three equal sides.

25. That an isosceles triangle, which has but two equal sides.

26. That a scalene triangle, which has three unequal sides ^p.

27. Moreover, amongst three-sided figures, that is a right-angled triangle, which has a right angle.

28. That an obtuse-angled triangle, which has an obtuse angle.

29. And that an acute-angled triangle, which has three acute angles.

30. Amongst four-sided figures, that is a square, whose sides are equal, and its angles right angles.

31. That an oblong which is right-angled, but not equal sided.

32. That a rhombus, which is equal-sided, but not right-angled.

33. That a rhomboides, whose opposite sides and angles are equal, but is neither equal-sided nor right-angled ^q.

34. All four-sided figures besides these may be called trapeziums.

35. Parallels are right lines, which being in the same plane, and produced infinitely either way, will not meet one another either way ^r.

^p This definition perhaps might as well have been omitted, for I see no great use thereof.

^q This definition of a rhomboides, and that before it of a rhombus, are of no great consequence.

^r Some call parallels right lines those in one plane, neither inclining nor reclining, but having all the perpendiculars equal, that are drawn from the points of either of the lines to the other line.—Others call them equidistant right lines in one plane.—Others say, they are such that tend to a point infinitely distant.—Others, that as a perpendicular is the shortest of all right lines drawn from a given point in a plane to a given right line in that plane, so the longest right line that can be drawn from that point to [or rather towards] that given right line, is parallel to it. But Euclid's definition of parallels is in general the best; though in some particular instances, these others are not without their use.—Parallels must both lie in the same plane, for otherwise, two right lines in different planes, the one in

POSTU-

POSTULATES or PETITIONS.

1. Grant that a right line may be drawn from any one point to another point.
2. That a finite right line may be continued directly forwards.
3. That a circle may be described from any centre to any distance.

COMMON NOTIONS or AXIOMS.

1. Things equal to the same thing, are equal to one another.
2. If equal things be added to equal things, the wholes are equal.
3. If equal things be taken away from equal things, the remainders are equal.
4. If equal things be added to unequal things, the wholes are unequal.
5. If equal things be taken away from unequal things, the remainders are unequal.
6. Things which are double to the same thing, are equal to one another.
7. Things which are halves of the same thing, are equal to one another.
8. Things which mutually agree with one another, are equal to one another^s.
9. The whole is greater than a part of it.
10. * All right angles are equal to one another.
11. If a right line falling upon two right lines does make the internal angles on the same side less than two right angles, those right lines, being infinitely produced, do meet

one plane above, and the other in a plane beneath, may be infinitely produced both ways, and never meet, and yet not be such as Euclid calls parallels.

^s This eighth axiom is universally convertible; although Proclus, Borellus, Taquet, &c. say it is not. — See Barrow's learned arguments upon this subject in his 13th Mathematical lecture.

* In some books the 10th and 11th axioms are reckoned amongst the Postulates.

on that side where the angles are less than two right angles^t.

12. Two right lines do not comprehend a space.

^t All these axioms are so evident, when the words by which they are declared, are understood, that no-body can even deny their truth if he would: But I think the word equal, should have been first defined; which is, that those things are equal, that actually or potentially possess the same space, or actually or potentially agree together; that is, actual or possible congruency is equality. And notwithstanding the fault that is generally found with the eleventh axiom, *viz.* that it is rather a demonstrable proposition than an axiom; I was always, and ever shall be of opinion, that it is so clear, when understood, that no demonstration is required to make it more evident; and really think, those who have demonstrated it, have been only trifling; and instead of making it more evident, have more obscured it.—Since two right lines *AB*, *CD* in the same



plane, must necessarily meet or be parallel; and if they meet, it must either be to the right or left of the right line *EF* falling upon them, what man of common sense would say they would meet to the left? If both the angles *E*, *F* on the right were less than both the angles *AEF*, *CFE* on the left. And since two different right lines cannot be drawn from the same point out of a given right line parallel to that given right line; and since it is demonstrated at prop. 28. lib. 1. If right lines which are parallel, that is, never meet, make the two inward angles on the same side of a right line, falling upon them, both together equal to two right angles, certainly and most evidently, when two such angles are less than two right angles, the right lines which are crossed by the third right line (with which they make these angles) will not be parallel, and so must necessarily meet. Wherefore those who will not have this to be an axiom, must at least, I think, grant it to be a corollary or consequence arising from prop. 28. lib. 1. requiring no other proof to make its evidence more clearly appear, than that proposition; and by means of such a corollary the 29th proposition may be demonstrated.

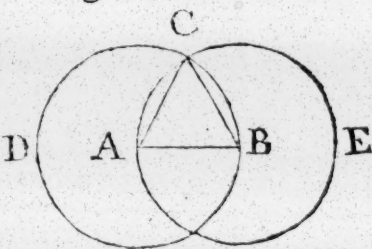
PROPO.

PROPOSITION I. PROBLEM.

To constitute an equilateral triangle upon a given finite right line.

LET AB be a given finite right line: it is required to constitute an equilateral triangle upon the right line AB .

From the centre A , with the distance AB , let the circle BCD be described [by postul. 3.] and again from the centre B , with the distance BA , let the circle ACE be described [by postul. 3.] and from the point C , in which the circles cut one another, let the right lines CA , CB be drawn to the points A and B , [by postul. 1.]



Therefore because the point A is the centre of the circle BCD , the right line AC [by def. 15.] will be equal to AB : Again, because the point B is the centre of the circle CAE , the right line BC will be equal to the right line BA . But it has been proved, that the right line AC is equal to the right line AB ; therefore each of the lines AC , CB , is equal to the right line AB . But things that are equal to one and the same thing, are also equal to one another [axiom 1.]. Therefore the right line AC is equal to the right line CB ; wherefore the three right lines AC , AB , BC are equal to one another.

And therefore the triangle ACB is equilateral [by def. 24.]. And also constituted upon the given finite right line AB . Which was to be done.

PROP. II. PROBL.

To put a right line at a given point, equal to a given right line.

Let the given point be A , and the given right line be BC : it is required to put a right line at the point A equal to the given right line BC .

Draw [by post. 1.] a right line AB from the point A to the point B ; and upon the same constitute [by prop. 1.]

of the right lines $A E$ and C is equal to the right line $A D$; wherefore the right line $A E$ is also equal to the right line C .

Therefore two unequal right lines, $A B$ and C , being given, there is taken from the greater $A B$, a line equal to C the lesser. Which was to be done.

v The manual operation of this and the last problem may be much shorter performed with a pair of compasses, or other such like instrument. But as these instrumental operations are no postulatums of Euclid's, they are therefore not to be admitted in the pure elements of geometry. Euclid himself, doubtless, very well knew that these problems might be mechanically resolved more easily by compasses; and even that he might have made postulatums, to put a given right line at a given point equal to a given right line, and to cut off a right line from a given right line equal to a given right line. But he thought it not right so to do, well knowing, that taking too many things for granted, not only lessens science, but may mislead and beget error. And, on the contrary, that geometrical constructions and demonstrations, even of propositions of no great apparent value in themselves, are always best to be given, were it for no other reason, than to beget a habit of exactness in the geometrician, and thereby a stronger disposition in him to avoid error, and obtain truth.

PROP. IV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have one angle of the one equal to one angle of the other, viz. that which is contained under the equal lines; then shall the base of the one be equal to the base of the other; and one triangle equal to the other triangle; and the remaining angles of the one shall be equal to the remaining angles of the other, each to each, which are opposite to the equal sides^w.

Let the two triangles be $A B C$, $D E F$, having the two sides $A B$, $A C$ of the one equal to the two sides $D E$, $D F$ of the other, each to each, viz. $A B$ to $D E$, and $A C$ to $D F$; and the angle $B A C$ of the one, equal to the angle $E D F$ of the other. I say, the base $B C$ is equal to the base $E F$, and the triangle $A B C$ shall be equal to the triangle $E D F$; and the remaining angles shall be equal to the remaining angles, each

each to each, which are opposite to the equal sides; that is, the angle ABC to the angle DEF , and the angle ACB to the angle DFE .

For if the triangle ABC be applied to the triangle DEF , and the point A be put upon the point D , and the right line AB upon the right line DE , then shall the point B agree with the point E ; because AB is equal to DE : But since the right line AB agrees with DE , the right line AC shall also agree with DF , because the angle BAC is equal to the angle EDF . Wherefore the point C will agree with the point F , because the right line AC is equal to the right line DF ; but the point B will agree with the point E . And therefore the base BC agrees with the base EF ; for if the point B agrees with the point E , and the point C with the point F ; but the base BC does not agree with the base EF ; it is necessary that two right lines must contain a space, which [by ax. 12.] is impossible. Therefore the base BC agrees with the base EF , and accordingly is equal to it. Therefore the whole triangle ABC agrees with the whole triangle DEF , and will be equal to the same; and the remaining angles will agree with the remaining angles, and therefore shall be equal to them, *viz.* the angle ABC to the angle DEF , and the angle ACB to the angle DFE .

If therefore two triangles have two sides of the one equal to two sides of the other, each to each; and have one angle of the one equal to one angle of the other, *viz.* that which is contained under the equal lines; then shall the base of the one be equal to the base of the other, and one triangle shall be equal to the other triangle; and the remaining angles of the one shall be equal to the remaining angles of the other, each to each, which are opposite to the equal sides. Which was to be demonstrated.

^w Some, for want of making a difference between a geometrical congruency and a mechanical one; that is, between an intellectual or mental one, and an actual sensual one made with the hands and the eyes, have taken occasion to find fault with the demonstration of this proposition; falsely affirming it to be only a mechanical demonstration, and not a geometrical one.

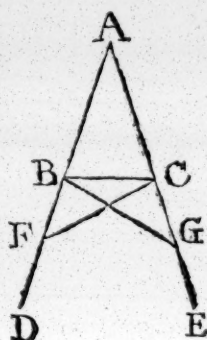
PROP.

PROP. V. THEOR.

The angles at the base of isosceles triangles are equal to one another; and the equal right lines being produced, the angles under the base shall be equal to one another^x.

Let ABC be an isosceles triangle, having the side AB equal to the side AC . And [by post. 2.] continue the equal sides AB , AC directly forwards to D and E : I say, the angle ABC is equal to the angle ACB ; and the angle CBD equal to the angle BCE .

For in the right line BD , let there be taken any point as F ; and from the greater line AE , let be taken AG equal to AF the less [by prop. 3.] and draw the right lines FC , GB .



Therefore because AF is equal to AG , and the right line AB equal to the right line AC ; there are two right lines FA , AC , equal to two right lines GA , AB , each to each, containing the common angle FAG : wherefore [by prop. 4.] the base FC will be equal the base GB ; and the triangle AFC equal to the triangle AGB ; and the remaining angles shall be equal to the remaining angles, each to each, which are opposite to the equal sides, *viz.* the angle ACF to the angle ABG , and the angle AFC to the angle AGB . But since the whole line AF is equal to the whole line AG , and AB is equal to AC ; therefore the remainder BF will be equal to the remainder CG [by ax. 3.] But it has been proved, that the right line FC is equal to the right line GB ; therefore there are two right lines BF , FC , equal to two right lines CG , GB , each to each; and the angle BFC equal to the angle CGB ; and the right line BC is their common base: wherefore the triangle BFC [by prop. 4.] shall be equal to the triangle CGB , and the remaining angles shall be equal to the remaining angles, each to each, which are opposite to the equal sides: Therefore the angle FBC is equal to the angle GCB , and the angle BCF to the angle GBC . Now since it has been demonstrated, that the whole angle ABG is equal to the whole angle ACF , and the angle CBG equal to

to the angle BCF ; the remaining angle ABC [by ax. 3.] will be equal to the remaining angle ACB ; and they are at the base of the triangle ABC . But also it has been demonstrated, that the angle FCB is equal to the angle GCB , and they are under the base.

Therefore the angles at the base of isosceles triangles are equal to one another, and the equal right lines being produced, the angles under the base shall be equal to one another. Which was to be demonstrated.

* Scarborough says, the latter part of this proposition seems not to be Euclid's, but put in by somebody else; because, says he, it is certain, that Euclid never laid down an elementary proposition useless in any part thereof; nor ever put that for one part of the proposition, which is only used as a means to prove the other. But herein I differ from Scarborough; for why might not Euclid himself demonstrate the latter part of his proposition first? If it should be asked, Why then he did not set it down first? I answer, Because the latter part was the less important and useful part of the proposition, and therefore the former part deserved the first place. And if a proposition be not all Euclid's, because it is not useful in all its parts; other propositions of these elements must not, in all their parts, be Euclid's; such as prop. 16. and 31. lib. 3. &c. But Scarborough is out here; for the latter part of this proposition is useful in the demonstration of one case of the 6th proposition; viz. when the point D falls within the triangle ACD ; as also one case of the 24th proposition, when the point F (see the figure of that proposition) falls within the triangle EDG .

PROP. VI. THEOR.

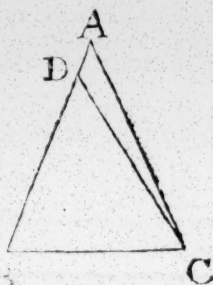
If two angles of a triangle be equal to one another, then shall the sides opposite to the equal angles be equal to one another.

Let the triangle ABC have the angle ABC equal to the angle ACB . I say, the side AC will be equal to the side AB .

For if AC be unequal to AB , one of them will be the greater. Let the greater be AB , and from the greater AB let be taken [by prop. 3.] the right line DB equal to the lesser AC , and draw the right line DC .

Then

Then because DB is equal to the right line AC , and the right line BC is a common side; there will be two right lines DB , BC equal to two right lines AC , CB , each to each, and the angle DBC is equal to the angle ACB : wherefore the base DC will be equal to the base AB , and the triangle ABC will be equal to the triangle DCB ; the greater to the less; which is absurd. Therefore the right line AB is not unequal to the right line AC ; they are consequently equal.



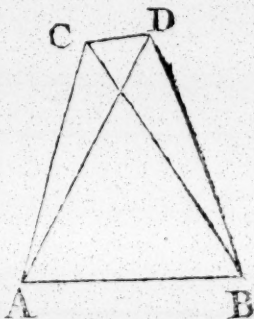
Therefore if two angles of a triangle be equal, the sides opposite to the equal angles will be equal. Which was to be demonstrated.

PROP. VII. THEOR.

Two right lines cannot be constituted on the same right line equal to two other right lines, each to each, at different points on the same side, and having the same ends with the first right lines.

For, if possible, let two right lines AD , DB be constituted upon the right line AB , equal each to each, to two right lines AC , CB , at different points C , D , situated on the same side the line AB , the lines AD , DB , having the same ends A , B with the two first lines AC , CB ; so that CA be equal to DA , both having the same end A : and CB be equal to DB , both having the same end B : for join the right line CD .

Then because AC is equal to AD , the angle ACD is equal to the angle ADC [by prop. 5.] wherefore the angle ADC is greater than the angle DCB , and the angle CDB is much greater than the angle DCB . Again, because the right line CB is equal to the right line DB , the angle CDB will be equal to the angle DCB . But it has been proved to be much greater, which is impossible.



Therefore two right lines cannot be constituted upon the same right line equal to two other right lines, each to each, at

at different points, on the same side, and having the same ends with the first right lines; which was to be demonstrated.

Although this 7th proposition may be thought to be of no great importance in itself, yet its use in demonstrating the 8th proposition makes it both valuable and necessary. It is as much one of the pillars to support the great elementary building of our author, as any the most beautiful ones of them all.

PROP. VIII. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also equal bases, then shall the angle, contained by the equal sides of the one triangle, be equal to the angle contained by the equal sides of the other^z.

Let there be two triangles ABC , DEF , having the two sides AB , AC of the one, equal to the two sides DE , DF of the other, each to each; that is, the side AB equal to DE , and the side AC equal to DF ; and let also the base BC of the one be equal to the base EF of the other. I say, the angle BAC is equal to the angle EDF .

For if the triangle ABC be applied to the triangle DEF , and the point B put upon the point E ; and the right line BC upon the right line EF ; the point C will agree with the point F ; because the right line BC is equal to the right line EF . But since the right line BC agrees with the right line EF ; the right lines BA , AC will also agree with the right lines ED , DF : for if the base BC agrees with the base EF , but the sides BA , AC do not agree with the sides ED , DF , but change their situation, as EG , GF , then upon the same right line can be constituted two right lines equal to two other right lines, each to each, at different points on the same side, and having the same ends with the first right lines. But [by prop. 7.] they cannot be so constituted. Therefore the base BC agreeing with the base EF , the sides BA , AC cannot disagree with the sides ED , DF ; they will therefore agree: and so the angle BAC will agree with the angle EDF , and be equal to it.

If

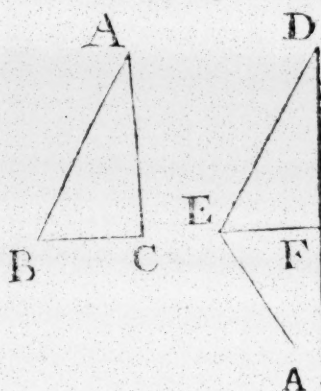
If therefore two triangles have two sides of the one equal to two sides of the other, each to each; and have also equal bases; then shall the angle contained under the equal right lines of the one, be equal to the angle contained under the equal right lines of the other; which was to be demonstrated.

^z Proclus gives the following direct demonstration of this theorem. Let the base BC be

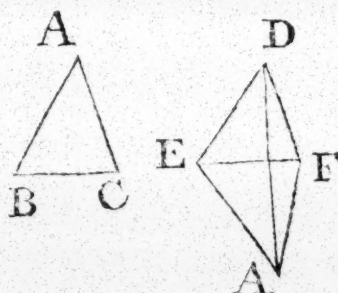
conceived to be put upon the base EF , so that the triangle ABC falls not upon the triangle EDF , but on the contrary way, as does the triangle AEF . Then the two sides DF, FA either both make one right line, *viz.* when the angles C, F are right angles, or not: if they both make one right line, as DA , the proposition is thus proved. Because in the triangle AED , the two sides AE, DE are put equal (for now the right line AE is the same with AB , which by the supposition is equal to the right line DE) the angles A and D upon the base AD , will be [by 5. 1.] equal to one another; which was to be demonstrated. Now if neither DF, FA , nor DE, EA , both make one right line, from D to A draw a right line DA ; which will either fall within the triangles or without them. First, let it fall within them, which will happen when the angles at E, F are acute. Then because in the triangle AED the two sides AE, ED are put equal, the two angles EAD, EDA at the base AD will [by 5. 1.] be equal to one another. By the same reason, since the two sides AF, DF are equal [by supposition] the two angles FAD, FDA upon the base DA will be equal. If therefore these equals be added to those, the whole angles EAF, EDF will [by axiom 2.] be equal; which was to be demonstrated.

Again, let the right line DA fall without the triangles, which will happen when the angles at F are obtuse; then because in the

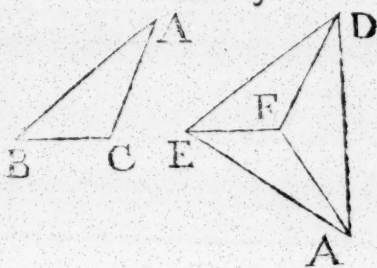
Case 1.



Case 2.



Case 3.



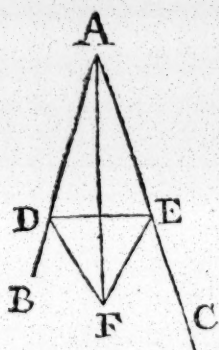
the triangle AED the two sides AE, DE are supposed to be equal. The angles EAD, EDA [by 5. 1.] will be equal. By the same reason, because the two sides AF, FD of the triangle AFD are equal, the angles FAD, FDA upon the base AD will be equal. These therefore being taken away from the former, and there will remain [by axiom 3.] the angles EAF, EDF equal; which was to be demonstrated.

PROP. IX. PROBL.

To bisect or cut a given right-lined angle into two equal parts^a.

Let the given right-lined angle be BAC ; it is required to bisect or cut the same into two equal parts.

Let there be taken any point as D in the right line AB , and from the right light line AC [by prop. 3.] cut off the right line AE equal to the right line AD ; and draw the right line DE ; and upon the right line DE [by prop. 1.] describe the equilateral triangle DEF , and draw the right line AF : I say, the angle BAC is cut into two equal parts by the right line AF .



For because the right line AD is equal to the right line AE , and AF is common, therefore the two sides DA, AF will be equal to the two sides EA, AF , each to each; and the base DF is equal to the base EF . Therefore the angle DAF is equal to the angle EAF [by prop. 8.]

Therefore the given right-lined angle BAC is cut into two equal parts by the right line AF ; which was to be done.

^a The operation would be somewhat more elegant, if an isosceles triangle DEF were described or constructed upon the right line DE , having one of its equal sides DF or EF equal to AD or AE . Hence it appears, how any given angle may be divided into 4, 8, 16, 32, &c. equal parts, viz. by bisecting each part again. But from Euclid's postulatus cannot be obtained the division generally of an angle into any other number of equal parts, as 3, 5, 6, 7, &c.

PROP. X. PROBL.

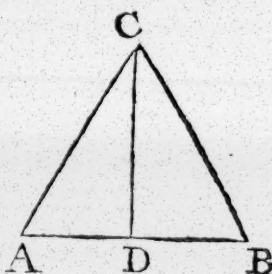
To cut a given finite right line into two equal parts^b.

Let AB be the given finite right line, which is to be cut into two equal parts.

Upon AB let there be made the equilateral triangle ABC [by prop. 1.] and cut the angle ACB into two equal parts [by prop. 9.] by the right line CD . I say, the right line AB is cut into two equal parts at D .

For because the side AC is equal to the side CB , and the side CD is common; the two sides AC, CD are equal to the two sides BC, CD , each to each, and the angle ACD is equal to the angle BCD : therefore the base AD [by prop. 4.] is equal to the base BD .

Therefore the given finite right line AB is cut into two equal parts in the point D ; which was to be done.



^b The solution of this problem is really no more than describing two equilateral triangles upon it, the one on one side the given right line AB , and the other on the other side that right line; and joining the angles of each of these triangles, which are opposite to the given right line AB , by a right line, which will cut the given right line AB into two equal parts.

PROP. XI. PROBL.

From a point given in a given right line, to draw a right line at right angles to it.

Let there be a given right line AB , and let the given point in the same be C . It is required to draw a right line from the given point C at right angles to the right line AB .

In the right line AC let there be taken any point D ; and [by prop. 3.] put the right line CE equal to the right line CD ; and [by prop. 1.] upon the right line DE describe the equilateral triangle FDE , and draw the right line FC . I say, from the given point C in the given right line AB is drawn the right line CF at right angles to AB .

For because the right line CD is equal to the right line CE , and the right line CF is common; there are two right lines DC, CF , equal to the two right lines EC, CF , each

C

to

to each, and the base DF is equal to the base EF : therefore the angle DCF is equal [by prop. 8.] to the angle ECF . And these are adjacent angles. But when a right line standing upon a right line makes the adjacent angles equal to one another, each of these equal angles is a right angle [by def. 10.] Therefore each of the angles DCF , ECF is a right angle.

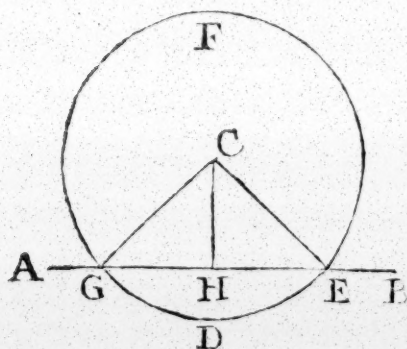
Therefore from a given point c , in a given line AB , the right line FC is drawn at right angles to the right line AB .

PROP. XII. PROBL.

From a given point, without a given infinite right line, to draw a right line perpendicular to it.

Let the given infinite right line be AB , and let the given point without the same be c ; it is required from the given point c without AB to draw a right line perpendicular to it.

Let any point D be taken on the other side of the line



AB ; and about the centre C , with the distance CD , let the circle EGF be described [by post. 3.] and let the right line EG be bisected in the point H [by prop. 10.] and draw the right lines CG , CH , CE : I say, that the right line CH is drawn from the given point c , without the given infinite right

line AB perpendicular to it.

For because GH is equal to the right line HE , and the right line HC is common; there are two right lines GH , HC , equal to two right lines EH , HC , each to each; and the base CG is equal to the base CE [by def. 15.] Therefore the angle CHG is equal to the angle $EH C$ [by prop 8.]. And these are adjacent angles. But when a right line standing upon a right line makes the adjacent angles equal to one another, each of the equal angles is a right angle, and the standing right line is called a perpendicular to that upon which it stands.

There-

Therefore upon the given infinite right line AB , from a given point C without it, is drawn the perpendicular CH to it: which was to be done.

^c It was very right in Euclid to suppose an infinite right line in this problem; for if it were a finite right line, a right line could not always be drawn from a given point out of it perpendicularly upon it.

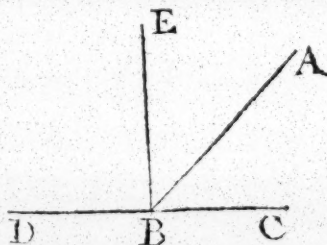
There are several practical ways of solving this problem, as well as the last, but the most expeditious of all is by a square.

PROP. XIII. THEOR.

If a right line standing upon a right line makes angles, these angles shall either be two right angles, or [both together] equal to two right angles^d.

For let any right line AB , standing upon the right line DC , make the angles CBA , ABD ; I say, these angles will either be right angles, or [both together] equal to two right angles.

For if the angle CBA be equal to the angle ABD ; then will they be both right angles [by def. 10.] but if not, let the right line EB be drawn from the point B at right angles to the right line DC [by prop. 11.] then the angles CBE ,



EBD are two right angles. And because the angle CBE is equal to both the angles CBA , ABE , if the common angle EBD be added, the angles CBE , EBD are equal to the three angles CBA , ABE , EBD . Again, since the angle DBA is equal to both the angles DBE , EBA , if the common angle ABC be added, the angles DBA , ABC will be equal to the three angles DBE , EBA , ABC . But the angles CBE , EBD have been proved to be equal to the same three angles; and things which are equal to the same thing, are equal to one another. Therefore the angles CBE , EBD are equal to the angles DBA , ABC ; but the angles CBE , EBD are two right angles. Therefore the angles DBA , ABC are equal to two right angles.

If therefore a right line standing upon a right line makes angles, these angles shall either be two right angles, or [both together] equal to two right angles: which was to be demonstrated.

^d This proposition seems to depend upon a certain axiom or common notion, *viz.* that so much as the angle ABD exceeds the right angle EBD , by so much is the remaining angle ABC exceeded by the right angle EBC ; for as there the angle ABE is the excess, so is the same angle ABE here the deficiency. Wherefore it necessarily follows, that both the angles ABD , ABC are equal to two right angles, the one being just as much above a right angle, as the other wants of one.

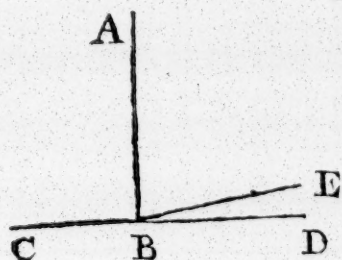
PROP. XIV. THEOR.

If at a point in any right line two right lines drawn contrary ways do make, with the first line, the adjacent angles equal to two right angles; those right lines will be directly placed to one another, or both fall into one right line^e.

For at the point B in the right line AB let two right lines BC , BD , drawn contrary ways, make, with AB , the adjoining angles ABC , ABD equal to two right angles. I say, the right lines BD , CB will lie directly to one another, or both fall into one right line.

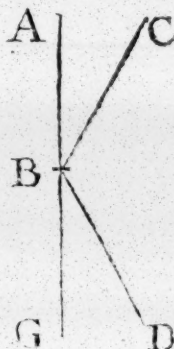
For if BD lies not in the same direction with BC , let [by post. 2.] CB , BE both lie in the same direction.

Then because the right line AB stands upon the right line CBE , the angles ABC , ABE are [by prop. 13.] equal to two right angles; but [by the supposition] the angles ABC , ABD are also equal to two right angles: Therefore the angles CBA , ABE are equal to the angles CBA , ABD . Take away the common angle ABC , then the remaining angle ABE will be equal to the remaining angle ABD , the less equal to the greater; which is impossible: therefore, the right lines BE , BC are not both in one direction. It is demonstrated after the same manner, that no right line but BD can have the same direction with the right line CB . Wherefore BC and BD lie both in the same right line.



If therefore at a point in any right line, two right lines, drawn contrary ways, do make with the first line the adjacent angles equal to two right angles; those right lines will be directly placed to one another, or both fall into one right line. Which was to be demonstrated.

* This proposition would not generally have held true, if the two right lines, as BC, BD had been both on the same side the right line AB ; for if the angle GBD , at the point B , made with the continuation BG of the right line AB , be equal to the angle ABC , made at the point B with the right line AB , the angles ABC, ABD taken together, are equal to two right angles. But the right lines BC, BD are far from coinciding; that is, both making one right line. Indeed, in this case, where the two right lines are both on one side the first proposed right line, the proposition never can take place, unless the angles ABC, ABD be both right angles.

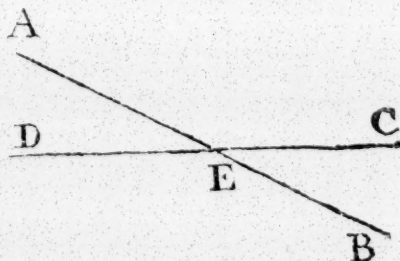


PROP. XV. THEOR.

If two right lines cut each other, they shall make the vertical angles equal to one another.

For let two right lines AB, CD cut each other in the point E . I say, the angle AEC is equal to the angle DEB , and the angle CEB equal to the angle AED .

For because the right line AE stands upon the right line CD , making the angles CEA, AED . Therefore the angles CEA, AED are [by prop. 13.] equal to two right angles. Again, because the right line DE stands upon the right line AB , making the angles AED, DEB ; the angles AED, DEB will be equal to two right angles. But the angles CEA, AED have been proved to be equal to two right angles. Therefore the angles CEA, AED are equal to the angles AED, DEB . Take away the common angle AED , then the remaining angle CEA is equal to the remaining angle DEB . In like manner it may



be proved, that the angle $\angle CEB$ shall be equal to the angle $\angle DEA$.

If therefore two right lines cut each other, they shall make the vertical angles equal to one another. Which was to be demonstrated.

Corol. From hence it is manifest, that if never so many right lines mutually cut one another in the same point, they shall make the angles at the point of intersection equal to four right angles.

PROP. XVI. THEOR.

Any one side of every triangle being produced, the outward angle is greater than either of the inward opposite angles.

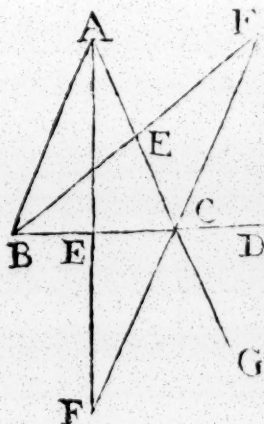
Let the triangle be BAC , and let the side BC be produced to D : I say, the outward angle $\angle ACD$ is greater than either of the inward opposite angles $\angle CBA$, $\angle BAC$.

For bisect [by prop. 10.] AC in the point E , and drawing the right line BE , produce it to F , and [by prop. 3.] put EF equal to BE , join FC , and continue out AC to G .

Then because AE is equal to EC , and BE equal to the right line EF ; there are two right lines AE , EB equal to two right lines CE , EF , each to each, the angle $\angle AEB$ also is equal [by prop. 15.] to the angle $\angle FEC$, for they are vertical angles. Therefore the base AB is equal to the base FC [by prop. 4.] and the triangle ABE equal to the triangle FEC , and the remaining angles equal to the remaining angles each to each, which are opposite to the equal sides: therefore the angle $\angle BAE$ is equal to the angle $\angle ECF$. But [by ax. 9.] the angle $\angle ECD$ is greater than the angle $\angle ECF$; therefore the angle $\angle ACD$ is greater than the angle $\angle BAE$. In like manner, if BC be bisected, it will be demonstrated that the angle $\angle BCG$, that is [by prop. 15.] the angle $\angle ACD$ is greater than the angle $\angle ABC$.

Therefore if any one side of every triangle be produced,

the



the outward angle is greater than either of the inward and opposite angles. Which was to be demonstrated.

PROP. XVII. THEOR.

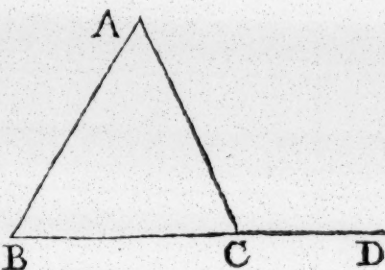
Any two angles of every triangle taken together, are less than two right angles.

Let there be a triangle ABC : I say, any two angles of the triangle ABC taken together, are less than two right angles.

For [by post. 2.] produce BC to D .

Then because the outward angle ACD of the triangle ACB , is [by prop. 16.] greater than the inward and opposite angle ABC ; add the common angle ACB ; then the angles BCD , ACB are greater than the angles ABC , BCA . But the angles BCD , ACB [by prop. 13.] are equal to two right angles: wherefore the angles ABC , BCA are less than two right angles. After the same way we demonstrate that the angles BAC , ACB are less than two right angles: as also the angles CAB , ABC .

Therefore any two angles of every triangle taken together, are less than two right angles. Which was to be demonstrated.



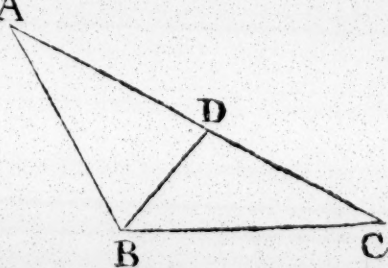
PROP. XVIII. THEOR.

The greater angle of every triangle is opposite to the greater side.

Let there be a triangle ABC , having the side AC greater than the side AB . I say, the angle ABC shall also be greater than the angle BCA .

For because the side AC is greater than the side AB , put [by prop. 3.] the right line AD equal to the side AB , and draw the right line BD :

Then because ADB is an outward angle of the triangle BDC , it is greater [by prop. 16.]



C 4

than

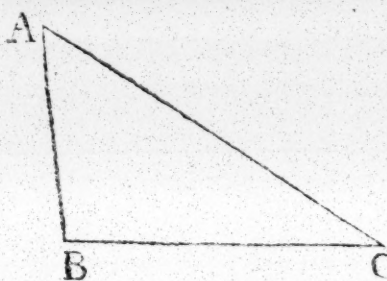
than the inward inward opposite angle DCB . But [by prop. 5.] the angle ADB is equal to the angle ABD , because the side AB is equal to the side AD : and therefore the angle ABD is greater than the angle ACB ; consequently the angle ABC is much greater than the angle ACB .

Therefore the greater angle of every triangle is opposite to the greater side. Which was to be demonstrated.

PROP. XIX. THEOR.

The greater side of every triangle is opposite to the greater angle.

Let there be a triangle ABC , having the angle ABC greater than the angle BCA . I say, the side AC is greater than the side AB .



For if it be not greater: the side AC is either equal to the side AB , or less than it. But the side AC is not equal to the side AB : for then the angle ABC would be equal [by prop. 5.] to the angle ACB . But it is not equal to it: therefore the

side AC will not be equal to the side AB . Neither, is the side AC less than the side AB : for then the angle ABC would be [by prop. 18.] less than the angle ACB . But it is not less: nor therefore will the side AC be less than the side AB . But it has been demonstrated not to be equal to it; wherefore the side AC is greater than the side AB .

Wherefore the greater side of every triangle is opposite to the greater angle. Which was to be demonstrated.

PROP. XX. THEOR.

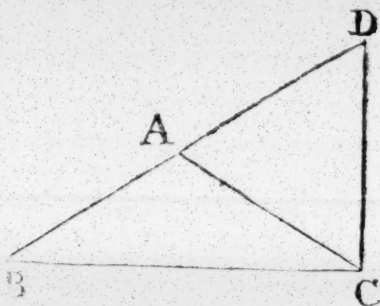
Any two sides of every triangle taken together, are greater than the remaining side.

For let there be a triangle ABC . I say, any two sides of this triangle taken together, are greater than the remaining side; that is, BA, AC together, greater than the side BC : AB, BC together greater than the side AC ; and BC, CA together greater than the side AB .

For

For produce the side BA to the point D , and [by prop. 3.] put DA equal to the right line CA , and join DC .

Then because DA is equal to the right line AC ; the angle ADC [by prop. 5.] will be equal to the angle ACD ; but [by ax. 9.] the angle BCD is greater than the angle ACD ; therefore the angle BCD is greater than the angle ADC . Also, because the triangle DCB has the angle BCD greater than the angle BDC , and [by prop. 19.] the greater side is opposite to the greater angle, the side DB will be greater than the side BC .



But the right line DB is equal to the sides AB , AC : therefore the sides BA , AC together, are greater than the side BC . After the same manner we demonstrate, that the sides AB , BC together, are greater than the side CA , and the sides BC , CA together, greater than the side AB .

Therefore any two sides of every triangle, taken together, are greater than the third side. Which was to be demonstrated.

^f Some think this proposition too self-evident to require a demonstration. But Euclid thought the fewest axioms, or indemonstrable propositions was best; and therefore has every where given demonstration when he could.

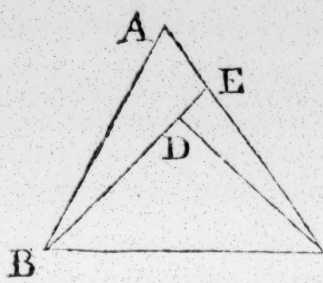
PROP. XXI. THEOR.

If from the ends of any side of a triangle, two right lines be constituted within the triangle; these lines shall indeed be less than the two remaining sides of the triangle, but will contain a greater angle.

For from the ends B , C , of one of the sides BC of the triangle ABC , let two right lines BD , CD be constituted within the triangle: I say the right lines BD , DC are less than the two remaining sides BA , AC of the triangle, but yet do contain an angle BDC greater than the angle BAC .

For continue out the right line BD to the point E .

Then because [by prop. 20.] the two sides of every triangle are greater than the third side remaining: the two sides AB , AE are greater than the side BE . Add the right line



line EC , which is common: therefore the sides BA , AC are greater than the right lines BE , EC . Again, because the two sides CE , ED of the triangle CED are greater than the side CD , let DB , which is common, be added: then the right lines CE , EB are greater than the right lines CD , DB . But the sides BA , AC have been proved to be greater than the right lines BE , EC : wherefore BA , AC are much greater than BD , DC .

Again, because the angle, which is without any triangle, is [by prop. 16] greater than either of the internal and opposite angles, the angle BDC , which is without the triangle CDE , is greater than the angle CED . By the same reason the angle CEB , which is without the triangle ABE , is greater than the angle BAC . But the angle BDC has been proved to be greater than the angle CEB : therefore the angle BDC is much greater than the angle BAC .

If therefore from the ends of any side of a triangle, two right lines be constituted within the triangle; these lines shall indeed be less than the two remaining sides of the triangle, but will contain a greater angle. Which was to be demonstrated.

PROP. XXII. PROBL.

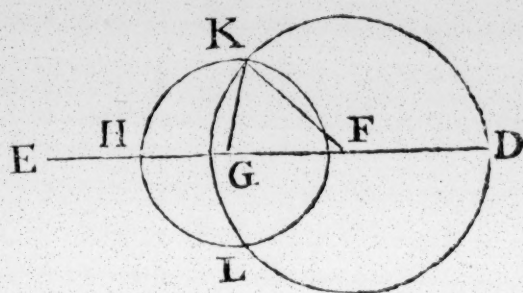
To constitute a triangle, whose sides shall be equal to three given right lines: but any two of these taken together, must be greater than that which remains.

Let the given right lines be A , B , C , whereof any two taken together, are greater than that remaining: viz. A and B greater than C ; also A and C greater than B ; and B and C , greater than A : it is required to constitute a triangle, whose three sides shall be equal to A , B , C .

Let a right line DE be put finite at D , but infinite towards E , and put [by prop. 3.] DF equal to the right line A , the right line FG equal to the right line B , and let GH be equal to the right line C , and about the centre F , with the distance FD , let a circle DKL be described [by post. 3.]

And

And again, about the centre G , with the distance GH , let a circle KLK be described, and draw the right lines KF , KG : I say, the triangle KFG is constituted of three right lines equal to the right lines A , B , C .



A —————
 B —————
 C —————

For because the point F is the centre of the circle DKL , the right line FD is [by def. 15.] equal to the right line FK : But FD is equal to the right line A : Therefore also the right line FK is equal to A . Again, Because the point G is the centre of the circle LKH , therefore GH is equal to the right line GK ; but GH is equal to the right line C . Therefore also GK is equal to the right line C , and FG is also equal to the right line B : wherefore the three right lines KF , FG , GK , are equal to the three right lines A , B , C .

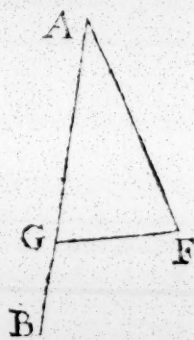
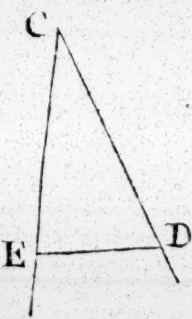
Therefore the triangle KFG is constituted, whose three sides KF , FG , GK are equal to the three given right lines A , B , C ; which was to be done.

PROP. XXIII. PROBL.

To make a right-lined angle at a given right line, and at a given point in it, equal to a given right-lined angle.

Let AB be the given right line, and A the point given therein; and let the right-lined angle given be DCE . It is required to make a right-lined angle at the given point A , with the line AB , equal to the given right-lined angle DCE .

In each of the right lines CD , CE take any two points D , E , and draw the right line DE , and of three right lines equal to the three right lines CD , DE , CE , constitute [by prop. 22.] a triangle AFG ; so that the right line CD be equal



to AF , the right line CE to AG , and the right line DE to the right line FG .

Then

Then because the two right lines DC , CE are equal to the two right lines FA , AG , each to each; and the base DE is equal to the base FG ; the angle DCE [by prop. 8.] is equal to the angle FAG .

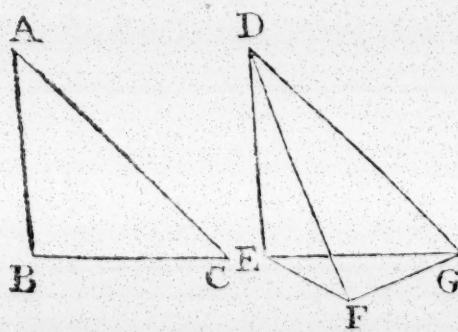
Therefore at the given right line AB , and at the given point A therein, is made the right-lined angle FAG , equal to the given right-lined angle DCE . Which was to be done.

§ There is an easier solution of this problem, by means of prop. 27. lib. iii. But as Euclid wanted this problem to help to demonstrate the 24th, he could not give that solution in this place, because it could not be here demonstrated; and so he gave the only one that could be demonstrated, upon the axioms and propositions already established. He has done the same all along. And whoever believes that Euclid was ignorant of shorter constructions of several of his problems that might be given, because he did not give them, I really think, are much mistaken; he rather omitted them because they could not be demonstrated but by propositions he was not yet come to, his constant purpose being to admit nothing without first demonstrating it.

PROP. XXIV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each; and the angle of the one contained under the equal right lines, greater than the angle of the other; then will the base of the one be greater than the base of the other.

Let there be two triangles ABC , DEF , having the two sides AB , AC equal to the two sides DE , DF , each to each; that is, the side AB equal to the side DE , and the



side AC equal to the side DF ; and let the angle BAC be greater than the angle EDF : I say, the base BC is greater than the base EF .

For because the angle BAC is greater than the angle EDF , at the right line DE , and at the point D , therein make [by prop. 23.] the angle EDG equal to the

the angle BAC ; and put DG [by prop. 3.] equal to either of the right lines AC , DF : also draw GE , FG .

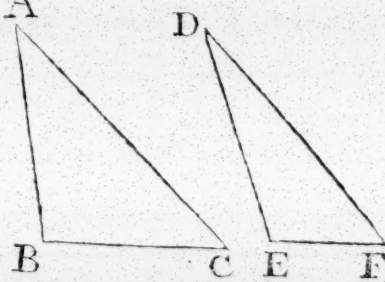
Now because AB is equal to the right line DE , and the right line AC equal to the right line DG ; therefore there are two right lines BA , AC equal to two right lines ED , DG , each to each; and the angle BAC [by constr.] is equal to the angle EDG : Therefore the base BC [by prop. 4.] will be equal to the base EG . Again, because DG is equal to the right line DF , and the angle DFG is [by prop. 5.] equal to the angle DGF ; the angle DFG will be greater than the angle EGF ; and therefore the angle EFG will be much greater than the angle EGF . And because the triangle EFG has the angle EFG greater than the angle EGF ; and [by prop. 19.] the greater side is opposite to the greater angle: the side EG is greater than the side EF . But [by constr.] the side EG is equal to the side BC ; and so the side BC is greater than the side EF .

If therefore two triangles have two sides of the one equal to two sides of the other, each to each; and the angle of the one contained by the equal right lines greater than the angle of the other; then will the base of the one be greater than the base of the other. Which was to be demonstrated.

PROP. XXV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each; and the base of the one be greater than the base of the other; they shall also have the angles contained by the equal sides, the one greater than the other^h.

Let there be two triangles ABC , DEF , having two sides AB , AC , equal to two sides DE , DF , each to each, viz. the side AB equal to the side DE , and the side AC equal to the side DF ; but the base BC greater than the base EF : I say, the angle BAC is greater than the angle EDF .



For if it be not greater, it will be either equal to, or less than it. But the angle BAC is not equal to the angle EDF ,

$\angle EDF$, for if it were [by prop. 4.] the base BC would be equal to the base EF . But it is not so [by supposition.] Therefore the angle BAC is not equal to the angle EDF , neither is it less; for if it were, the base BC [by prop. 24.] would be less than the base EF : But it is not; therefore the angle BAC is not less than the angle EDF . But it has been proved not to be equal to it neither. Wherefore the angle BAC is greater than the angle EDF .

If therefore two triangles have two sides of the one equal to two sides of the other, each to each; and the base of the one be greater than the base of the other; they shall also have the angles contained by the equal sides, the one greater than the other; which was to be demonstrated.

^h This proposition is demonstrated directly by Menelaus of Alexandria, and by Hero. See Proclus's Commentary upon the first book of Euclid; wherein are extant their demonstrations. Clavius has also translated them into Latin.

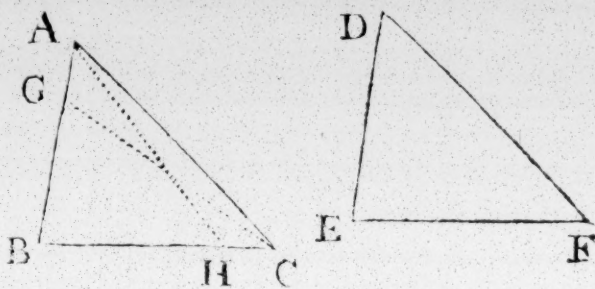
PROP. XXVI. THEOR.

If two triangles have two angles of the one equal to two angles of the other, each to each; and one side of the one equal to one side of the other, either that side which is between the equal angles, or that which is opposite to one of them; then will the remaining sides of the one triangle be equal to the remaining sides of the other, each to each; and the remaining angle of the one will be equal to the remaining angle of the other.

Let there be two triangles ABC , DEF , having the two angles ABC , BCA of the one equal to the two angles DEF , EFD of the other, each to each, viz. the angle ABC equal to the angle DEF , and the angle BCA equal to the angle EFD : also let one side of the one be equal to one side of the other. And first, Let these sides be BC , EF , lying between the equal angles: I say also, the remaining sides of the one will be equal to the remaining sides of the other, each to each, viz. the side AB equal to the side DE , and the side AC to the side DF ; and the remaining angle BAC equal to the remaining angle EDF .

For

For if the side AB be unequal to the side DE , one of them will be the greatest. Let AB be the greater, and [by prop. 3.] put the right line BG equal to ED , and join GC .



Then because the side BG is equal to the side DE , and the side BC to the side EF , viz. the two sides BG, BC equal to the two sides DE, EF , each to each; and the angle GBC is equal to the angle DEF : therefore will the base GC be equal to the base DF [by prop. 4.], the triangle GCB will be equal to the triangle DEF , and the remaining angles will be equal to the remaining angles which are opposite to the equal sides. Therefore the angle GCB is equal to the angle DFE . But the angle DFE is put equal to the angle BCA : wherefore the angle BCG is equal to the angle BCA , the less equal to the greater, which is impossible: therefore the side AB is not unequal to the side DE ; and so it is equal to it. But the side BC is equal to the side EF , and the two sides AB, BC are equal to the two sides DE, EF , each to each; and the angle ABC is equal to the angle DEF . Wherefore the base AC [by prop. 4.] will be equal to the base DF , and the remaining angle BAC is equal to the remaining angle EDF .

Secondly, Let the sides AB, DE opposite to the equal angles be equal; that is, the side AB to the side DE . I say, the remaining sides of the one triangle will be equal to the remaining sides of the other; that is, the side AC equal to the side DF , and BC equal to the side EF ; and also the remaining angles BAC and EDF equal.

For if BC be unequal to EF , one of them is the greater, which let be BC (if possible.) And put [by prop. 3.] BH equal to EF , and join AH .

Then because the side BH is equal to the side EF , and the side AB equal to the side DE ; viz. the two sides AB, BH equal to the two sides DE, EF , each to each; and the angles contained by them equal: therefore the bases AH, DF [by prop. 4.] are equal. And the triangle ABH is equal to the triangle DEF , and the remaining angles equal

equal to the remaining angles, each to each; which are opposite to the equal sides: Therefore the angle BHA is equal to the angle EFD . But [by supposition] the angle EFD is equal to the angle BCA : therefore the angle BHA is equal to the angle BCA ; viz. the outward angle BHA of the triangle AHC equal to the inward opposite angle BCA ; which [by prop. 16.] cannot be. Therefore the side BC is not unequal to the side EF , and so it is equal to it. But AB is equal to the side DE : wherefore two sides AB, BC are equal to two sides DE, EF , each to each; and they contain equal angles: Therefore the base AC [by prop. 4.] is equal to the base DF , and the triangle ABC is equal to the triangle DEF , and the remaining angle BAC equal to the remaining angle EDF .

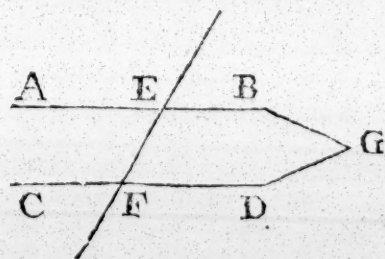
If therefore two triangles have two angles of the one equal to two angles of the other, each to each; and one side of the one equal to one side of the other; either that which is between the equal angles, or that which is opposite to one of them; the remaining sides of the one triangle will be equal to the remaining sides of the other, each to each; and the remaining angle of the one will be equal to the remaining angle of the other. Which was to be demonstrated.

PROP. XXVII. THEOR.

If a right line falling upon two right lines makes the alternate angles equal to one another, these right lines will be parallel to one another.

For let the right line EF , falling upon the right lines AB, CD , make the alternate angles AEF, EFD equal to one another. I say, the right line AB is parallel to the right line CD .

For if it be not parallel, the lines AB, DC produced will meet either towards BD or AC ; let them be produced towards BD , and meet in the point G .



Now the angle AEF being the external angle of the triangle EFG , is greater [by prop. 16.] than the inward opposite angle

EFG .

BFG: But [by the supposition] it is likewise equal to it; which is impossible: Therefore **AB**, **CD** continued out towards **BD** will not meet. After the same manner we demonstrate, that these lines will not meet towards **AC**: but right lines, which being either way produced, and do not meet, are [by def. 35.] parallel to one another. Therefore **AB** is parallel to **CD**.

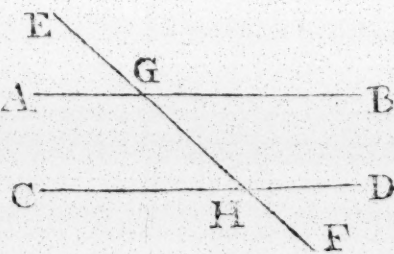
Wherefore if a right line falling upon two right lines makes the alternate angles equal to one another, these two right lines shall be parallel. Which was to be demonstrated.

PROP. XXVIII. THEOR.

If a right line falling upon two right lines makes the outward angle of the one equal to the inward opposite angle of the other, on the same side; or the inward angles on the same side together equal to two right angles: these right lines shall be parallel to one another.

For let the right line **EF**, falling upon the two right lines **AB**, **CD**, make the outward angle **EGB** equal to the inward opposite angle **GHD**, on the same side; or the inward angles **BGH**, **GHD**, on the same side together equal to two right angles. I say, the right line **AB** is parallel to the right line **CD**.

For because the angle **EGB** is equal to the angle **GHD**, and [by prop. 15.], the angle **EGB** equal to the angle **AGH**: the angle **AGH** also shall be equal to the angle **GHD**: they are alternate too. Therefore [by prop. 27. 1.] **AB** is parallel to **CD**.



Secondly, Because the angles **BGH**, **GHD** are equal to two right angles, and [by prop. 13.] the angles **AGH**, **BGH** also equal to two right angles; the angles **AGH**, **BGH** will be equal to the angles **BGH**, **GHD**. Take away the common angle **BGH**, and then the remaining angle **AGH** is equal to the remaining angle **GHD**; and they are alternate angles: Therefore **AB** [by prop. 27.] is parallel to **CD**.

D

If

If therefore a right line falling upon two right lines makes the outward angle of the one equal to the inward opposite angle of the other, on the same side; or the inward angles, on the same side, together equal to two right angles; these right lines shall be parallel to one another. Which was to be demonstrated.

PROP. XXIX. THEOR.

If a right line falls upon two parallel right lines, it makes the alternate angles equal to one another; the outward angle equal to the inward and opposite angle on the same side; and the inward angles on the same side together equal to two right angles¹.

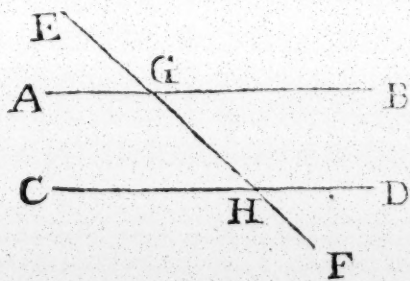
For let the right line EF fall upon the parallel right lines AB , CD : I say, it makes the alternate angles AGH , GHD equal to one another; the outward angle EGB equal to the inward opposite angle GHD on the same side; and the two inward angles BGH , GHD on the same side together equal to two right angles.

For if the angle AGH be unequal to the angle GHD , one of them is the greater, which let be AGH . Then because the angle AGH is greater than the angle GHD , put the common angle BGH to both. Therefore will the angles AGH , BGH be greater than the angles BGH , GHD .

But the angles AGH , BGH are [by prop. 13.] equal to two right angles: Therefore the angles BGH , GHD are less than two right angles. But [by ax. 11.] right lines infinitely produced from angles less than two right angles, will meet: Therefore the right lines AB , CD infinitely produced, will meet one another: but they do not meet, since they are supposed to be parallel. Therefore the angle AGH is not unequal to the angle GHD . It is therefore equal to it.

Secondly, The angle AGH is equal [by prop. 15.] to the angle EGB : therefore also EGB will be equal to GHD .

Thirdly,

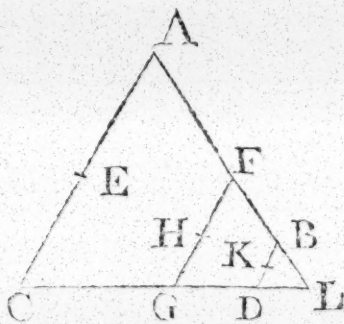


Thirdly, Put BGH , which is common, to both; then the angles EGB , BGH are equal to the angles BGH , GHD . But EGB , BGH together, are [by prop. 13.] equal to two right angles: Therefore also BGH , GHD are, together, equal to two right angles.

Therefore if a right line falls upon two parallel right lines, it will make the alternate angles equal; the outward angle equal to the inward and opposite angle on the same side; and the inward angles, on the same side, together equal to two right angles. Which was to be demonstrated.

ⁱ Scarborough, in his annotations upon this proposition, towards the end, has advanced a strange paradox, *viz.* that two right lines drawn from angles less than two right angles, may in some manner be for ever prolonged, yet shall they never meet together; and (as he thinks) has given a demonstration thereof. But he has mistook the matter. Let us see what he says.

Let two right lines AB , CD be cut by AC , making the inward angles BAC , DCA less than two right angles. Now let AC be cut into halves, or otherwise, in E : and equal to EA let be put AF , and to EC , CG ; then draw FG . Again, let FG be cut in half in H , and equal to HF let be put FB , and to HG , GD ; then draw BD : I say, that



the lines AB , CD may for ever be thus prolonged, yet never shall they meet together. For, if possible, let them meet in the point L ; therefore BD being cut in half in K , the line KB shall be equal to BL , and KD to DL . Wherefore of the triangle BDL , the sides DL , LB shall be equal to the third side BD ; which is impossible [by 20. 1.]. Therefore the lines AB , CD , drawn from angles less than two right, may for ever be prolonged and never meet together; which was to be demonstrated — These are Scarborough's own words. Now, I say, he is deceived here; for his two right lines AB , CD will really meet in L , neither AL nor CL being infinite; and his ultimate triangle BDL will become infinitely small, the points B , D , L all coinciding together: so that here the *both* proposition becomes unapplicable. And I think, from all that he says, we can obtain no more than this, *viz.* that an infinite number of right lines FG , BD , all shorter and shorter, and nearer and nearer approaching, may be drawn within the triangle ACL ; and all the prolongations that AB , CD will acquire, by an infinite

number of particular operations of his, will be only the finite lengths BL , DL .

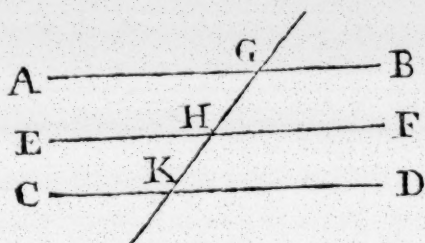
PROP. XXX. THEOR.

Those right lines that are parallel to the same right line, are parallel to one another.

Let each of the right lines AB , CD be parallel to the right line EF : I say also, AB will be parallel to CD .

For let the right line GK fall upon them.

Then because the right line GK falls upon the parallel



right lines AB , EF , the angle AGH [by prop. 29.] is equal to the angle GHE . Again, because the right line GK falls upon the parallel right lines EF , CD , the angle GHE is [by prop. 29.]

equal to the angle GKD . But it has been proved, that the angle AGK is also equal to the angle GHE : and therefore the angle AGK will be equal to the angle GKD ; and they are alternate angles: Therefore [by prop. 27.] the right line AB is parallel to CD .

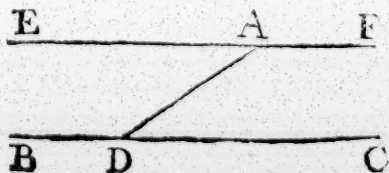
Wherefore those right lines that are parallel to the same right line, are parallel to one another; which was to be demonstrated.

PROP. XXXI. PROBL.

To draw a right line through a given point, parallel to a given right line.

Let the given point be A , and the given right line be BC : It is required to draw a right line through the given point A , parallel to the given right line BC .

Take any point D in BC , and join AD ; and [by prop. 23.] with the right line AD , at the point A therein, make the angle DAE equal to the angle ADC , and continue out the right line EA to F .



Then

Then because the right line AD falling upon two right lines EF, BC makes the alternate angles EAD, ADC equal; the right line EF will be parallel to the right line BC [by prop. 27.].

Therefore the right line EAF is drawn through the given point A , parallel to the given right line BC . Which was to be done.

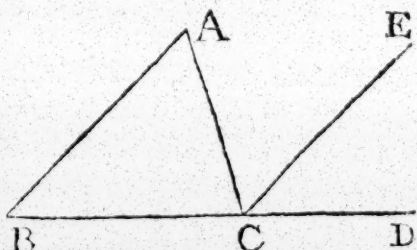
PROP. XXXII. THEOR.

If one side of any triangle be produced, the outward angle is equal to the inward opposite angles taken together; and the three inward angles of every triangle altogether are equal to two right angles^k.

Let ABC be a triangle, and one of its sides BC produced to D . I say, the outward angle ACD is equal to the two inward opposite angles CAB, ABC taken together; and the three inward angles ABC, BCA, CAB altogether are equal to two right angles.

For [by prop. 31.] draw the right line CE through the point C parallel to the right line AB .

Then because AB is parallel to CE , and the right line AC falls upon them; the alternate angles BAC, ACE [by prop. 29.] are equal to one another. Again, because AB is parallel to CE , and the right line BD falls upon them; the outward angle ECD is equal to the inward opposite angle ABC . But it has been proved, that the angle ACE is equal to the angle BAC : wherefore the whole outward angle ACD is equal to both the inward opposite angles BAC, ABC .



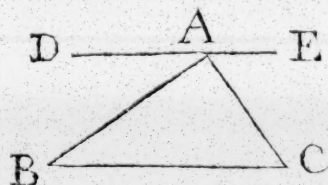
Let the angle ACB , which is common, be added to both; then the angles ACD, ACB , together, are equal to all the three angles ABC, BCA, CAB . But [by prop. 13.] both the angles ACD, ACB are equal to two right angles: Therefore all the angles ACB, CBA, CAB are equal to two right angles.

Therefore if one side of any triangle be produced, the outward angle is equal to both the inward opposite angles;

and all the three inward angles of every triangle are equal to two right angles. Which was to be demonstrated.

^k Eudemus says, the Pythagoreans first of all demonstrated, that the three angles of a triangle are equal to two right angles, after the following manner :

Let there be a triangle ABC , and through the point A [by 31. 1.] draw the right line DE parallel to BC . Then because [by 29. 1.] the alternate angles DAB , ABC are equal to one another, if the equal angles EAC , ACB be added (for these are alternate angles); the two angles DAB , EAC [by axiom 2.] will be equal to the two angles ABC , ACB . And so, adding the common angle BAC , the three angles DAB , BAC , CAE will be equal to the three angles ABC , BAC , ACB . But the angles DAB , BAC , CAE are equal to two right angles; as is evident [by prop. 13. 1.]. Therefore the three angles ABC , BAC , ACB are equal to two right angles; which was to be demonstrated.



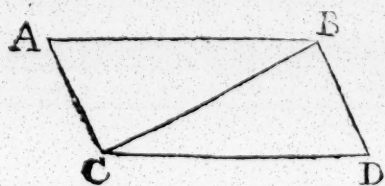
PROP. XXXIII. THEOR.

Two right lines, which join two equal and parallel right lines, towards the same parts, are also themselves equal and parallel¹.

Let the right lines AC , BD join towards the same parts the equal and parallel right lines AB , CD : I say, AC , BD are equal and parallel.

For join BC .

Then because AB is parallel to CD , and the right line



BC falls upon them; the alternate angles ABC , BCD [by prop. 29.] are equal. Again, because AB is equal to CD , and BC is common, the two sides AB , BC are equal to the two

sides CD , BC , and the angle ABC equal to the angle BCD : Therefore [by prop. 4.] the base AC is equal to the base BD , and the triangle ABC equal to the triangle BCD . And the remaining angles of the one triangle will be equal to the remaining angles of the other, each to each, which are

are opposite to the equal sides. Therefore the angle ACB is equal to the angle CBD . And because the right line BC , falling upon the two right lines AC , BD , makes the alternate angles ACB , CBD equal to one another; the right line AC [by prop. 27.] is parallel to the right line BD . It has also been proved to be equal to it.

Therefore two right lines which join two equal and parallel right lines, towards the same parts, are themselves both equal and parallel. Which was to be demonstrated.

¹ If two right lines do not the same way join the equal parallel right lines, that is, do not join the answerable extremes A , C , and B , D , of the equal and parallel right lines AB , CD ; but the alternate ones A , D ; B , C , crossing one another. The latter part of the proposition will never be true, and the former one but seldom; therefore Euclid was obliged to say, that the right lines must join the equal and parallel ones the same way.

This proposition of Euclid is more extensive than it is generally thought to be. For it neither confines the number of the right lines joining the equal and parallel right lines to two, nor the equal parallels to two, though the demonstration does.

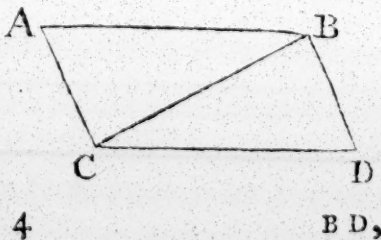
I also think, the proposition, as generally taken, might have been more clearly expressed thus: If the extremes of two equal and parallel right lines be joined by the extremes of two other right lines, not so as that these lines cut one another: these right lines will also be equal and parallel.

PROP. XXXIV. THEOR.

The opposite sides and angles of any parallelogram or figure bounded by parallel lines, are equal to one another, and a diameter bisects it^m.

Let there be a parallelogram $ACDB$, and its diameter BC : I say, the opposite sides and angles of the parallelogram $ACDB$ are equal, and a diameter divides it into two equal parts.

For because AB is parallel to CD , and the right line BC falls upon them. The alternate angles ABC , BCD [by prop. 29] are equal to one another. Again, because AC is parallel to



BD , and the right line BC falls upon them, the alternate angles ACB , CBD are equal to one another: Therefore there are two triangles ABC , CBD , which have two angles ABC , BCA equal to two angles CBD , CBD of the other, each to each; and one side of the one equal to one side of the other, *viz.* the side BC between the equal angles, which is common to both. Therefore [by prop. 26.] the remaining sides shall be equal to the remaining sides, each to each; and the remaining angle to the remaining angle. Wherefore the side AB is equal to the side CD , and the side AC to BD , and the angle BAC to the angle BDC . And because the angle ABC is equal to the angle CBD , and the angle CBD equal to the angle ACB ; the whole angle ABD will be equal to the whole angle ACD . It has been also demonstrated, that the angle BAC is equal to the angle BDC .

Therefore the opposite sides and angles of any parallelogram [or four-sided figure bounded by parallel lines] are equal.

I say also, the diameter bisects it. For because AB is equal to CD , and BC is common; the two sides AB , BC are each equal to the two sides DC , CB , and the angle ABC is equal to the angle CBD : Therefore the base AC [by prop. 4.] is equal to the base BD ; and so the triangle ABC will be equal to the triangle CBD .

Therefore the diameter BC cuts the parallelogram $ACDB$ into two equal parts. Which was to be demonstrated.

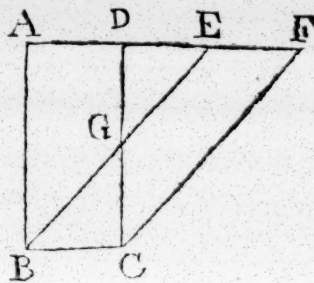
^m Some have thought, that Euclid should have particularly defined a parallelogram at the beginning of his Elements; but he did not think fit to do it; because, perhaps, he had not a mind to unnecessarily increase the number of his definitions. In effect, he himself has told us in the proposition what it is, *viz.* a parallel-lined space, i. e. a space contained under parallel lines, *paralleloi* signifying, in Greek, parallels, and *gramma* a line: so that a Grecian, at least, could scarcely be at a loss to know the signification of the word parallelogram, after he knew what were lines, and what were parallels. But whether it be so right to leave out the word space, as it is done all along afterwards throughout the whole Elements, I cannot well tell, unless it be for the sake of brevity, to call parallelogram spaces only parallelograms.

PROP. XXXV. THEOR.

Parallelograms constituted upon the same base, and between the same parallels, are the one equal to the other.

Let the parallelograms $ABCD$, $EBCF$ be constituted upon the same base BC , and between the same parallels AF , BC : I say, the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

For because $ABCD$ is a parallelogram, [by prop. 34.] AD is equal to BC ; and for the same reason EF also is equal to BC : wherefore AD is also equal to EF , and DE is common to both. Wherefore the whole AE



is [by ax. 2.] equal to the whole DF . But AB [by prop. 34.] is equal to DC ; therefore the two sides EA , AB of the triangle EAB are equal to the two sides FD , DC of the triangle $FD C$, each to each; and the angle FDC [by prop. 29.] equal to the angle EAB , the outward one to the inward one. Therefore [by prop. 4.] the base EB is equal to the base FC , and the triangle EAB equal to the triangle $FD C$. If the triangle DGE , which is common to both, be taken away; there will remain the trapezium $ABGD$ [by ax. 3.] equal to the trapezium remaining $EGCF$. If to both these be added the common triangle BGC , the whole parallelogram $ABCD$ will be equal to the whole parallelogram $EBCF$.

Therefore parallelograms constituted upon the same base, and between the same parallels, are the one equal to the other. Which was to be demonstrated.

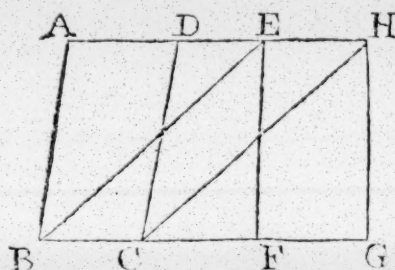
PROP. XXXVI. THEOR.

Parallelograms constituted upon equal bases, and between the same parallels, are equal the one to the other.

Let the parallelograms $ABCD$, $EFGH$ be constituted upon the equal bases BC , FG , and between the same parallels

railels AH , BG . I say, the parallelograms $ABCD$, $EFGH$ are equal to one another.

For join BE , CH . Then because BC is equal to FG , and FG equal to EH ; therefore will BC be equal to EH . They are parallel too, and BE , CH joins them. But right lines which join equal and parallel right lines towards the same parts, are equal and parallel too [by prop. 33.]:



Therefore EB , CH are both equal and parallel: and so $EBCH$ is a parallelogram, which [by prop. 35.] is equal to the parallelogram $ABCD$; for it has the same base BC , and is constituted between the same parallels, BC , AH . By the like way of reasoning, the parallelogram $EFGH$ is equal to this parallelogram $EBCH$: therefore the parallelogram $ABCD$ is equal to the parallelogram $EFGH$ too.

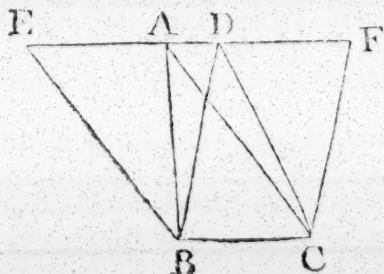
Wherefore parallelograms constituted upon equal bases, and between the same parallels, are equal the one to the other. Which was to be demonstrated.

PROP. XXXVII. THEOR.

Triangles constituted upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC , DBC be constituted upon the same base BC , and between the same parallels AD , BC : I say, the triangle ABC is equal to the triangle DBC .

For continue out AD both ways to the points E and F ; and through B [by prop. 31.] draw the right line BE parallel to the right line CA ; and through C draw CF parallel to BD .



Then $EBCA$, $DBCF$ are both parallelograms, and the parallelogram $EBCA$ [by prop. 35.] is equal to the parallelogram $DBCF$, for they stand both upon the same base BC , and are between the same parallels BC , EF . But the triangle

angle ABC is the one half of the parallelogram $EBCA$; since the diameter AB cuts it into halves, and the triangle DBC the one half of the parallelogram $DBCF$; because [by prop. 34.] the diameter DC cuts it into halves. But the halves of equal things are themselves equal: therefore the triangle ABC is equal to the triangle DBC .

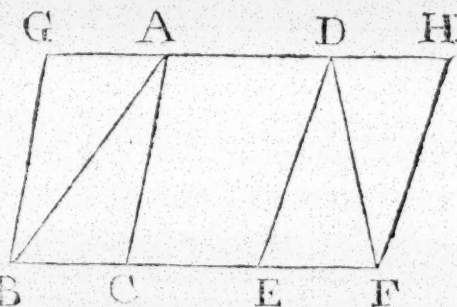
Therefore triangles constituted upon the same base, and between the same parallels, are the one equal to the other. Which was to be demonstrated.

PROP. XXXVIII. THEOR.

Triangles constituted upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC , DEF be constituted upon the equal bases BC , EF , and between the same parallels BF , AD : I say, the triangle ABC is equal to the triangle DEF .

For continue out AD both ways to the points G , H , and through B [by prop. 31.] draw BG parallel to CA , but through F draw FH parallel to DE .



Then $GBCA$, $DEFH$ are both parallelograms.

But [by prop. 36.] $GBCA$ is equal to $DEFH$, for they stand upon equal bases BC , EF , between the same parallels BF , GH . But the triangle ABC is one half of the parallelogram $GBCA$, because [by prop. 34.] the diameter AB cuts it in halves; and the triangle FED is the one half of the parallelogram $DEFH$; for the diameter DF cuts it into halves. But the halves of equal things are themselves equal. Therefore the triangle ABC is equal to the triangle DEF .

Wherefore triangles constituted upon equal bases, and between the same parallels, are equal to one another. Which was to be demonstrated.

PROP.

PROP. XXXIX. THEOR.

Equal triangles, constituted towards the same parts, upon the same base, are between the same parallels.

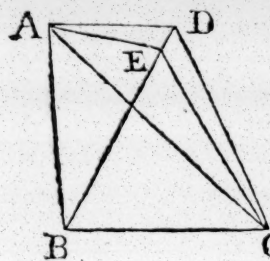
Let the equal triangles ABC , DBC be constituted upon the same base BC towards the same parts: I say, they are between the same parallels. For draw AD : I say, this is parallel to BC .

For if not, through the point A [by prop. 31.] draw the right line AE parallel to the right line BC , and draw EC .

Then [by prop. 37.] the triangle ABC is equal to the triangle EBC : for they are both upon the same base BC , and between the same parallels BC , AE . But [by supposition] the triangle ABC is equal to the triangle DBC . Therefore also the triangle DBC is equal to the triangle EBC , the greater equal to the less; which is impossible:

therefore AE is not parallel to BC . After the same manner we demonstrate, that no other line except AD is parallel to BC . Therefore AD is parallel to BC .

Wherefore equal triangles constituted towards the same parts, upon the same base, are between the same parallels. Which was to be demonstrated.



PROP. XL. THEOR.

Equal triangles constituted towards the same parts, upon equal bases, are between the same parallels.

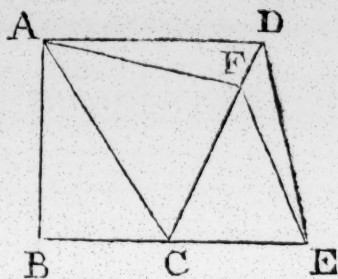
Let the equal triangles ABC , CDE be constituted upon equal bases BC , CE , towards the same part: I say, they are also between the same parallels. For draw AD : I say, AD is parallel to BE .

For if it be not, through A draw FA parallel to BE , and draw FE .

Then [by prop. 38.] the triangle ABC is equal to the triangle FCE ; for they are constituted upon equal bases BC , CE , and are between the same parallels BE , AF . But the triangle ABC is equal to the triangle DCE . And

so

so the triangle DCE shall be equal to the triangle FCE , the greater equal to the lesser; which is impossible. Therefore AF is not parallel to BE . After the same manner we demonstrate, that there is no other line except AD , parallel to BE . Wherefore AD will be parallel to BE .



Therefore equal triangles constituted upon equal bases, towards the same parts, are between the same parallels. Which was to be demonstrated.

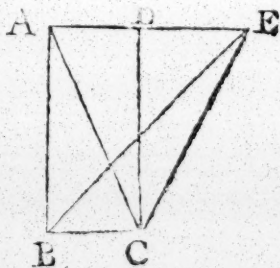
PROP. XLI. THEOR.

If a parallelogram and a triangle have the same base, and are between the same parallels; the parallelogram will be double to the triangle.

For let the parallelogram $ABCD$, and the triangle EBC have the same base BC , and be between the same parallels BC, AE : I say, the parallelogram $ABCD$ will be double to the triangle EBC .

For draw AC .

Then the triangle ABC [by prop. 37.] is equal to the triangle EBC ; for it is constituted upon the same base BC , and between the same parallels BC, AE . But the parallelogram $ABCD$ is double to the triangle ABC ; for the diameter AC [by prop. 34.] cuts it into halves: wherefore the parallelogram $ABCD$ will be double to the triangle EBC .



If therefore a parallelogram and a triangle have the same base, and are between the same parallels, the parallelogram will be double to the triangle. Which was to be demonstrated.

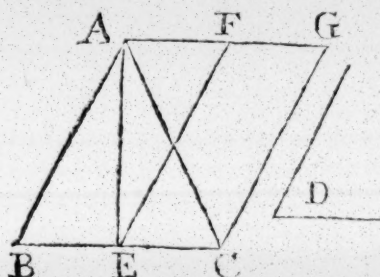
PROP. XLII. PROBL.

To constitute a parallelogram equal to a given triangle, and having one of its angles equal to a given right-lined angle.

Let the given triangle be ABC , and the given right-lined

lined angle D , it is required to constitute a parallelogram equal to the triangle ABC , having one of its angles equal to the given right-lined angle D .

Bisect BC [by prop. 10.] in E , and draw AE ; and with the right line EC , at the point



therein [by prop. 23.] make the angle CEF equal to the angle D . And [by prop. 31.] through A , draw AG parallel to EC ; and through C , draw CG parallel to FE : then is $FECG$ the parallelogram required.

For because BE is equal to EC , the triangle ABE [by prop. 38.] will be equal to the triangle AEC ; for they stand upon equal bases BE , EC , and are between the same parallels BC , AG . Therefore the triangle ABC is double to the triangle AEC . But [by prop. 41.] the parallelogram $FECG$ is double to the triangle AEC ; for it has the same base, and is between the same parallels: Therefore [by ax. 6.] the parallelogram $FECG$ is double to the triangle ABC , and has the angle CEF equal to the given angle D .

Therefore the parallelogram $FECG$ is made equal to the triangle ABC , having the angle CEF equal to the given angle D . Which was to be done.

PROP. XLIII. THEOR.

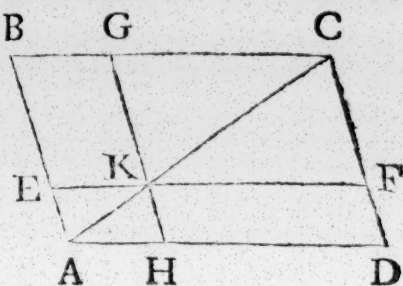
In every parallelogram, the complements of those parallelograms, which are about a diameter, are equal to one anotherⁿ.

Let there be a parallelogram $ABCD$, whose diameter is AC ; and let the parallelograms EH , FG be about the same: then BK , KD are called the complements of them. I say, the complement BK is equal to the complement KD .

For because $ABCD$ is a parallelogram, and AC is its diameter, the triangle ABC [by prop. 34.] is equal to the triangle ADC . Again, because $EKHA$ is a parallelogram, whose diameter is AK , the triangle AEK will be equal to the triangle AKH . For the same reason the triangle KFC is equal to the triangle KGC . Therefore since the triangle

AEK

$\triangle A E K$ is equal to the triangle $\triangle A H K$, and the triangle $\triangle K F C$ to the triangle $\triangle K G C$; the triangle $\triangle A E K$, together with the triangle $\triangle K G C$, is equal to the triangle $\triangle A H K$, together with the triangle $\triangle K F C$. But the whole triangle $\triangle A B C$ is also equal to the whole triangle $\triangle A D C$: Therefore the complement $\triangle B K$ remaining, is [by ax. 3.] equal to the complement $\triangle K D$ remaining.



Therefore in every parallelogram the complements of those parallelograms which are about a diameter, are equal to one another. Which was to be demonstrated.

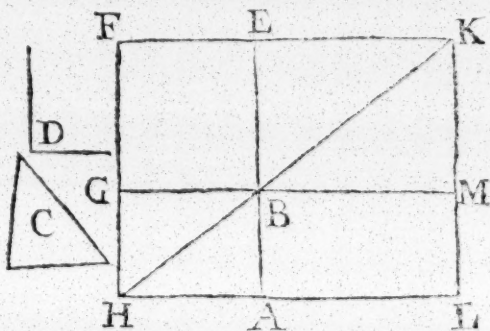
" It may, perhaps, be as well to express this proposition thus: If two right lines be drawn through a point in the diagonal of any parallelogram parallel to the sides, thereby making four lesser parallelograms within the greater, those two of the four lesser parallelograms, which fall without that diagonal, will be equal to one another.

PROP. XLIV. PROBL.

To apply * a parallelogram to a given right line equal to a given triangle, and having one angle equal to a given right-lined angle.

Let the given right line be $A B$, the given triangle c , and the given right-lined angle D . It is required to apply a parallelogram to the given right line $A B$, equal to the given triangle c , having one angle equal to D .

Constitute [by prop. 42.] the parallelogram $B E F G$ equal to the triangle c , having one angle $E B G$ equal to the angle D ; and put $B E$ in a right line with $A B$. Produce $F G$ to H , and through A draw [by prop. 31] $A H$ parallel to $B G$, or $E F$. Join $H B$. Then because



* That is, to make such a parallelogram, that a given right line shall be one of its sides; one of its angles shall be equal to a given right-lined angle, and the parallelogram shall be equal to a given triangle.

the

the right line HF falls upon the parallels AH , EF , the angles AHF , HFE [by prop. 29.] are equal to two right angles; and so BHG , GFE are less than two right angles. But right lines infinitely produced from angles less than two right angles [by ax. 11.] will meet one another. Let them be produced [by post. 2.] meeting at K , and thro' K draw [by prop. 31.] KL , parallel to EA or FH . And continue out AH , GB to the points L , M .

Then is $HLKF$ a parallelogram, whose diameter is HK , and the parallelograms about HK are AG , ME ; and these which are called the complements are LB , BF : therefore [by prop. 43.] the parallelogram LB is equal to BF . But BF is equal to the triangle C : wherefore LB will be equal to the triangle C too. And because [by prop. 15.] the angle GBE is equal to the angle ABM , and also GBE is equal to the angle D ; the angle ABM will be therefore equal to the angle D .

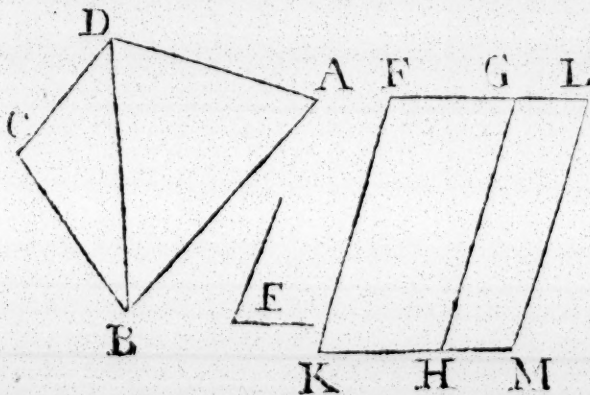
Wherefore at the given right line AB is applied the parallelogram LB equal to the triangle C , and having one angle ABM equal to the angle D . Which was to be done.

PROP. XLV. PROBL.

To constitute a parallelogram equal to a given right-lined figure, having one of its angles equal to a given angle.

Let the given right lined figure be $ABCD$, and the given right-lined angle be E : It is required to constitute a parallelogram equal to the given right-lined figure $ABCD$, having one angle equal to the angle E .

Draw DB , and constitute [by prop. 42.] the parallelo-



gram FH equal to the triangle ADB , having one angle HKF equal to the angle E . Then to the right line GH apply the parallelogram GME equal to the triangle DBC , having one angle GHM equal to

to the angle E . Then because the angle E is equal to either HKF , or GHM ; the angle HKF will be equal to the angle GHM . Let KHG be added to both; then the angles FKH , KHG are equal to the angles KHG , GHM . But the angles FKH , KHG [by prop. 29.] are equal to two right angles. Therefore the angles KHG , GHM will also be equal to two right angles. And so because at the given point H , in the right line GH , two right lines KH , HM not lying the same way, make the adjacent angles equal to two right angles; therefore [by prop. 14.] KH and HM make but one right line. And because the right line GH falls upon the parallels KM , FG , the alternate angles MHG , HGF [by prop. 29.] are equal. Add HGL to both: then are the angles MHG , HGL equal to the angles HGF , HGL . But [by prop. 29.] MHG , HGL are equal to two right angles: wherefore also the angles HGF , HGL will be equal to two right angles. Wherefore FG and GL are both in the same right line. And because KF is equal to HG , and parallel to it; and HG also to ML ; KF will be [by ax. 1. and prop. 30.] equal and parallel to ML . But the right lines KM , FL join them: therefore the right lines KM , FL [by prop. 33.] are equal and parallel: wherefore $KFLM$ is a parallelogram. And because the triangle ABD is equal to the parallelogram HF , and the triangle ABC to the parallelogram GM , the whole right-line figure $ABCD$ will be equal to the whole parallelogram $KFLM$.

Therefore the parallelogram $KFLM$ is constituted equal to the given right-lined figure $ABCD$, having one angle FKM , which is equal to the given angle E . Which was to be done.

PROP. XLVI. PROBL.

To describe a square upon a given right line.

Let AB be the given right line upon which it is required to describe a square.

From the given point A in the right line AB [by prop. 11.] draw the right line AC at right angles to AB ; and [by prop. 3.] put AD equal to AB . From the point D [by prop. 31.] draw DE parallel to AB ; and from the point B draw BE parallel to AD .

E

Then

Then is $ADEB$ a parallelogram: and so DE is equal to AB , and AD to BE : But AB is also equal to AD . Therefore the four right lines BA, AD, DE, EB are equal to one another. Wherefore the parallelogram $ADEB$ is equilateral: I say, it is also right-angled. For because the right line AD falls upon the two parallels AB, DE , the angles BAD, ADE [by prop. 29.] are equal to two right angles. But [by constr.] BAD is a right angle:

Therefore will ADE be a right angle too. But [by prop. 34.] the opposite sides and angles of any parallelogram are equal: wherefore each of the opposite angles ABE, BED is a right angle; and so $ADEB$ is right-angled. It has also been proved to be equilateral. Therefore it is necessarily a square, and is described upon the right line AB . Which was to be done.

° Some have thought that Euclid should have demonstrated that the squares described upon equal lines are equal; and that if the squares are equal, the lines upon which they are described are also equal. But he thought there was no occasion for this, it being too evident to require a demonstration; for if the squares are supposed to be laid upon one another, they will agree together, and so will all their sides and angles; that is, they will be all equal.

Proclus demonstrates it more at large; but I think he might as well have let it alone.

PROP. XLVII. THEOR.

In right-angled triangles, the square described upon the side opposite to the right angle, is equal to both the squares described upon the sides containing the right angle^p.

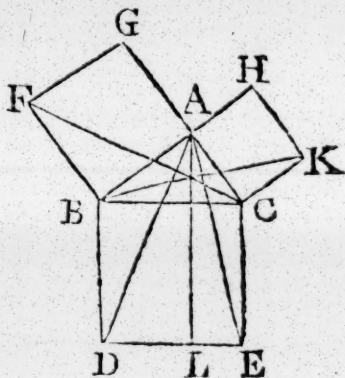
Let ABC be a right-angled triangle, having the right angle BAC . I say, the square described upon the right line BC , is equal to both the squares described upon the sides BA, AC .

For upon BC describe the square $BDEC$; and upon BA, AC , the squares GB, HC ; and through A draw AL parallel to BD or CE , and draw AD, FC .

Then

Then because the angles BAC , BAG are each right angles; and two right lines AC , AG not lying the same way, make the adjacent angles, at the point A therein, equal to two right angles; the right lines CA , AG are in the same direction. And for the

same reason AB and AH make but one right line. And because the angle DBC [by ax. 10.] is equal to the angle FBA , for they are both right angles. Add the angle ABC , which is common; then will the whole angle DBA be equal to the whole angle FBC .



But because the two sides DB , BA are equal to the two sides CB , BF , each to each; and the angle DBA is equal to the angle FBC ; the base AD [by prop. 4.] will be equal to the base FC ; and the triangle ABD to the triangle FBC . But [by prop. 41.] the parallelogram BL is double to the triangle ABD ; for they have the same base BD , and are between the same parallels BD , AL : but the square GB is double to the triangle FBC , since they have the same base FB , and are between the same parallels FB , GC . But the doubles of equal things are themselves equal: therefore the parallelogram BL is equal to the square GB . After the same manner, by drawing AE , BK , we demonstrate, that the parallelogram CL is also equal to the square HC . Therefore the whole square $BDEC$ is equal to both the squares GB , HC . And $BDEC$ is the square described upon the right line BC ; but the squares GB , HC are described upon the sides BA , AC . Therefore the square described upon the side BC , is equal to both the squares described upon the sides BA , AC .

Therefore in right-angled triangles, the square described upon the side opposite to the right angle, is equal to both the squares described upon the sides containing the right angle. Which was to be demonstrated.

^p This famous, useful, and elegant proposition, is reckoned to have Pythagoras for its inventor. It may be demonstrated several other ways. See Clavius, Schouten, and others; but there is none so simple and elegant as that of Euclid. The proposition is only a particular case of prop. 31. lib. 6.

As by this proposition it is easy to find a right line whose

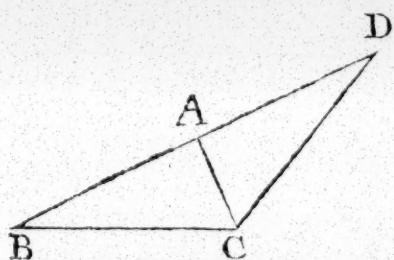
square shall be equal to the squares of two given right lines, so it is as difficult to find a right line, whose cube shall be equal to the cubes of two given right lines. Indeed, by a right line and circle this is impossible to do at all. But, granting the description of the conic sections, or the way of finding two mean proportionals between two given right lines, it may be done with tolerable ease.

PROP. XLVIII. THEOR.

If the square described upon one side of a triangle be equal to both the squares described upon the other two sides of that triangle; the angle contained by these two remaining sides will be a right angle.

For let the square described upon one side BC of the triangle ABC be equal to both the squares described upon the remaining sides BA, AC of that triangle: I say, the angle BAC is a right angle.

For [by prop. 11.] draw AD from the point A at right angles to AC . Put AD equal to BA , and draw DC .



Then because DA is equal to AB ; the square described upon DA will be equal to the square described upon AB .

Add the common square described upon AC . Then the squares described upon DA, AC are equal to the squares described upon BA, AC . But [by prop. 47.] the square described upon DC is equal to both the squares described upon DA, AC , for the angle DAC is a right one: but the square described upon BC is put equal to both the squares described upon BA, AC . Therefore the square described upon DC is equal to the square described upon BC ; and so the side DC is equal to the side BC . And because AD is equal to AB , and AC is common, the two sides AD, AC are equal to the two sides BA, AC , and the base DC is equal to the base CB . Therefore [by prop. 8.] the angle DAC is equal to the angle BAC . But the angle DAC is a right angle: Wherefore BAC will be a right angle too.

If therefore the square which is described upon one side of a triangle be equal to both the squares described upon the other two sides, the angle contained by these other two sides will be a right angle. Which was to be demonstrated.

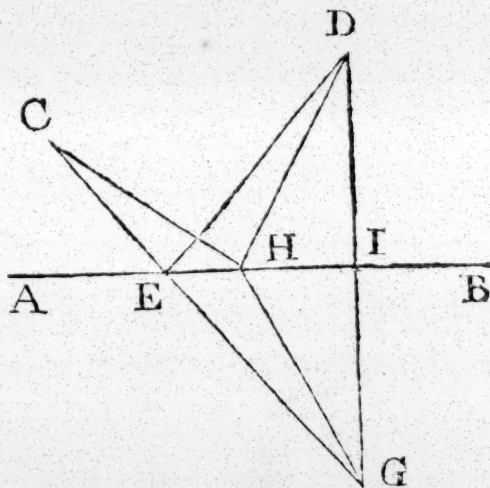
Additions

Additions to the First Book.

PROP. I. THEOR.

If a right line and two points on the same side of it be given, and two right lines be drawn from those points to any point in the given right line; these two right lines taken together will be the least of all, when they make equal angles with the given right line.

Let the given right line be AB , and the given points on the same side of it C, D , and let the point E be so taken in the given right line AB , that drawing two right lines CE, DE to it from the given points C, D , the angles AEC, BED which they make with AB , be equal to one another: I say, the right lines CE, DE taken together, will be less than any other two right lines CH, DH taken together, drawn from the given points C, D to any other point H taken in the right line AB .



For from one of the given points, as D , [by 12. 1.] draw the right line DI perpendicular to AB , which continue out to meet CE continued at G . Join CH, GH . Then because the vertical angles AEC, GEB [by 15. 1.] are equal to one another, and the angle DEB [by supposition] is equal to the angle AEC ; therefore the angles DEB, GEB are equal to one another. And because [by constr.] DI is perpendicular to AB , the angle AID is a right angle. And so will the angle AIG be a right angle. Wherefore the two triangles EDI, GEI have two angles DEI, DIE equal to two angles GEI, GIE , each to each, and the

side $E I$ common. Therefore [by 26. I.] the rest of their sides will be equal, and the remaining angles. Wherefore the side $E D$ will be equal to the side $E G$, and the side $I D$ to the side $I G$; and adding $C E$ to both, $C E$, $E D$ together will be equal to $C E$, $E G$ together; that is, to $C G$. Again, because the right angle $E I D$ is equal to the right angle $E I G$, the side $D I$ equal to the side $I G$, and the side $H I$ common; the triangles $D H I$, $G H I$ [by 4. I.] will have their sides $D H$, $H G$ equal: and adding $C H$ to each of them, $D H$, $C H$ together, will be equal to $C H$, $H G$ together. But [by 20. I.] $C G$ is less than $C H$, $H G$ together; that is, since $C G$ has been proved to be equal to $C E$, $E D$, and $C H$, $H G$ equal to $C H$, $H D$; $C E$, $E D$ together will be less than $C H$, $H D$ together.

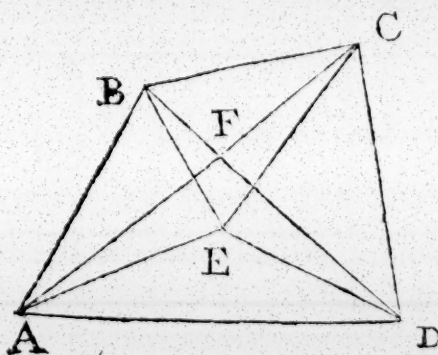
Therefore if a right line and two points on the same side of it be given, and two right lines be drawn from those points to any point in the given right line; these two right lines taken together will be the least of all, when they make equal angles with the given right line. Which was to be demonstrated.

PROP. II. THEOR.

The two diagonals of any right-lined quadrilateral figure, are together less than the four right lines that are drawn from any point whatsoever to the four angles of that figure [except it be the intersection of the diagonals.]

Let $A B C D$ be a right-lined quadrilateral figure, and the right lines $A C$, $B D$ its diagonals, intersecting one another in F .

Let E , be any assumed point [different from F] and let



the four right lines $A E$, $B E$, $C E$, $D E$ be drawn from E to the angles A , B , C , D of the quadrilateral figure $A B C D$: I say, the diagonals $A C$, $B D$ taken together are less than the four right lines $A E$, $B E$, $C E$, $D E$ taken together.

For since [by 20. I.] the two sides of any triangle, howso-

howsoever taken, are greater than the third side, the side AC of the triangle ACE is less than the sides AE, EC . So likewise the side BD of the triangle BED is less than the sides BE, ED ; therefore the sides AC, BD put together, are less than the sides AE, EC, BE, ED put together; that is, the two diagonals AC, BD together, are less than the four right lines AE, EC, BE, ED together.

Therefore the two diagonals of any right-lined quadrilateral figure, are together less than the four right lines that are drawn from any point whatsoever to the four angles of that figure [except it be the intersection of the diagonals.] Which was to be demonstrated.

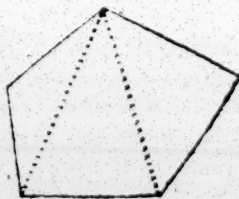
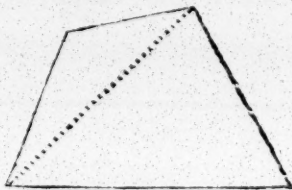
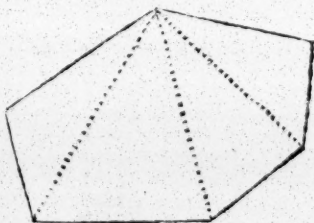
SCHOLIUM.

It may be demonstrated after the same manner, that any three sides of any quadrilateral figure, are greater than the third side; and that all the four sides are greater than twice the diagonals.

PROP. III. THEOR.

All the angles of any right-lined figure, when the angles are all inward, are equal to twice as many right angles, wanting four, as that figure has sides.

For any right-lined figure, by drawing diagonals, may be resolved or divided into as many triangles as the figure has sides, wanting two. As if a right-lined figure has four sides, it may be divided into two triangles; if a figure has five sides, it may be divided into three triangles; if it has six sides, it may be divided into four triangles, and so on. But since [by 32. 1.] the angles of all these triangles are equal to twice as many right angles as there are triangles, every triangle having two, and the angles of all the triangles are equal to all the inward angles of the right-lined figure; therefore all the inward angles of the figure are equal to twice



as many right angles as there are triangles; that is, equal to twice as many right angles, wanting the angles of two triangles; that is, wanting four right angles, as the figure has sides.

Therefore all the angles of any right-lined figure, when the angles are all inward, are equal to twice as many right angles, wanting four, as that figure has sides. Which was to be demonstrated.

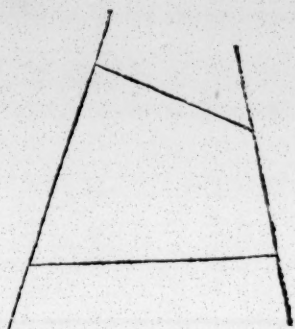
Otherwise thus:

From any point within the figure draw right lines to all the angles of the figure, which will resolve or divide the figure into as many triangles as the figure has sides. Then since all the angles of every triangle [by 32. 1.] are equal to two right angles, the angles of all the triangles together will make twice as many right angles as the figure has sides: therefore all the angles of the figure, together with all the angles about the assumed point within the figure, make twice as many right angles as the figure has sides. But because [by 13. 1.] two right lines mutually cutting one another, make four right angles at the point of section, and these four angles are equal to any number of angles about that point: for every whole is equal to all its parts taken together: therefore all the angles of a right lined figure, together with four right angles, are equal to twice as many right angles as the figure has sides. And taking away four right angles from each, there will remain all the angles of the figure equal to twice as many right angles as the figure has sides, wanting four right angles.

Therefore all the angles of any right-lined figure, when the angles are all inward, are equal to twice as many right angles, wanting four, as that figure has sides. Which was to be demonstrated.

Corol. 1. Hence, if the sides of any right-lined figure be continued out, all the angles without the figure made by each side, with the continuation of that next to it, will be equal to four right angles. For since each inward angle, and its correspondent outward one here spoken of, is equal to two right angles, all the angles of the figure, together with these outward angles, are equal

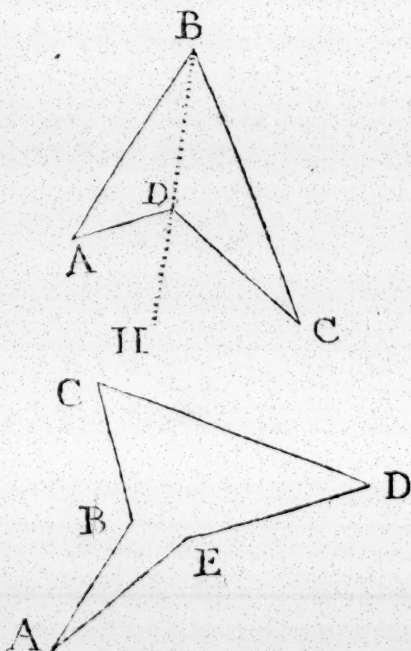
equal to twice as many right angles as the figure has angles, that is, as many sides. But since all the inward angles together, with four right angles, has been proved to be equal to twice as many right angles as the figure has sides; therefore all the inward angles, together with four right angles, is equal to all the inward and outward angles here spoken of, and taking away from both all the inward angles, there will remain all the outward angles here spoken of equal to four right angles.



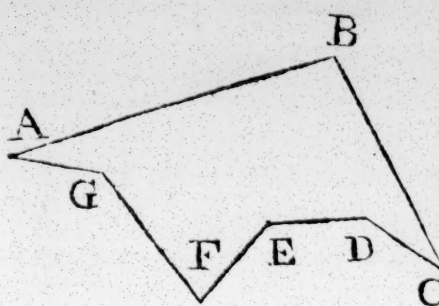
Corol. 2. Hence all right-lined figures of the same species have the sums of all their inward angles equal; and so are the sums of all their angles spoken of in the first corollary. Moreover, if the number of the sides of the several species of right-lined figures beginning at the triangle runs on according to the natural numbers 3, 4, 5, 6, 7, &c. the answerable numbers of the right angles equal to the sums of all the inward angles of each species of figures, will be the even numbers, 2, 4, 6, 8, 10, &c.

SCHOLIUM.

If a right-lined figure has outward angles as well as inward ones, the proposition does not then hold true. As in the quadrilateral figure *ABCD*, which has one outward angle *D*, as well as three inward ones *A*, *B*, *C*; or in the irregular pentagon *ABCDE*, having two outward angles *B*, *E*, and three inward angles *A*, *C*, *D*; or in the irregular heptagon *ABCDEFG*, having three outward angles *G*, *E*, *D*, and four inward ones *A*, *B*, *C*, *F*, &c. unless any one should object, that this distinction I have made between an inward and outward angle is unnecessary



sary



sary, and that he means by the inward angle of a figure, only the sum of two or more angles terminating at that angle; as, for example, the angle D of the quadrilateral figure ABCD, which is made up of the angles ADB, CDB.

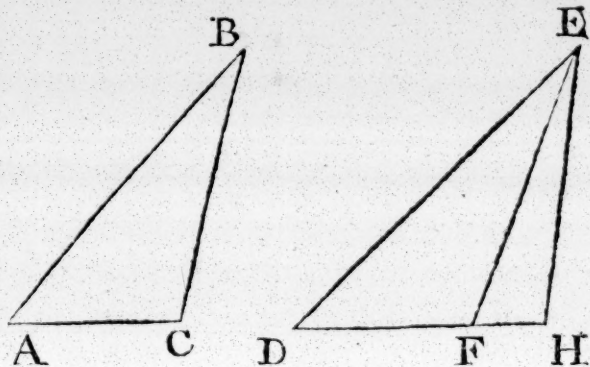
To this I answer, that the sum of those two angles do not properly constitute the angle D of the figure, but rather the sum of the angles ADH, CDH; because [by def. 8. 1.] the angle at D, made by the right lines AD, CD meeting in the point D, is the inclination [or opening] or leaning of the right lines AD, CD towards one another, and not their deviation or falling back or away from each other. Consequently, the angle at D of the quadrilateral figure ABCD, is, according to Euclid's definition of an angle, the angle ADC without the figure made by the inclination of its sides AD, CD; and so are the angles B, E of the pentagon, or the angles D, E, G of the heptagon. And these must be taken to be so, till we have a more general and distinct definition of a right-lined angle. This being admitted, in any right-lined figure having both inward and outward angles, the sum of all the outward angles is greater than the sum of all the inward angles, by as many right angles as there are units in the product of the multiplication of the number of sides by the number of the outward angles, added to 4, and lessened by twice the number of sides of the figure, as, if the figure has four sides, it can have one outward angle. And in this case, the outward angle is equal to all the three inward angles. If a pentagon has two outward angles and three inward ones, the sum of all the outward angles will be greater than the sum of all the inward angles, by four right angles: and so of others.

PROP. IV. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, and one angle of the one equal to one angle of the other, but not that angle contained under the equal sides; and if
the

the angles of each triangle opposite to one of the equal sides be either both acute or both obtuse : I say, the remaining side of the one triangle will be equal to the remaining side of the other ; and the remaining angles of the one equal to the remaining angles of the other, each to each.

Let ABC, DEF be the two triangles, having two sides AB, BC equal to two sides DE, EF , each to each ; and the angle BAC of the one equal to the angle EDF of the other, not being the angles contained under the equal sides. And if both the other angles ACB, DFE , opposite to the equal sides



AB, DE , be either an acute or an obtuse angle : I say, the remaining sides AC, DF will be equal ; as also the remaining angles ACB, DFE will be equal, as also ABC and DEF .

First, Let the equal angles BAC, EDF be acute, and the angles ACB, DFE be both obtuse.

For if the sides AC, DF of the triangles ABC, DEF be not equal, one of them must be the greater. Let the side AC be greater than the side DF ; and continuing out the side DF to H , take the side DH equal to the side AC , and join the right line EH .

Then because the sides AB, AC of the triangle ABC , are equal to the sides DE, DH of the triangle DEH , each to each ; and the angles BAC, EDH contained under them are equal. Therefore [by 4. 1.] the remaining angles ACB, DHE , and ABC, DEH , of the triangles ABC, DEF , will be equal ; as also the remaining sides BC, EH . Wherefore since the side BC [by supposition] is equal to the side EF , the side EF will be equal to the side EH : therefore the angles EFH, EHF [by 5. 1.] will be equal to one another. But because [by supposition] the angle

ACB

ACB or $D FE$ is obtuse; and since [by 13. 1.] the angles $D FE$, $E FH$ are equal to two right angles, the angle $E FH$ will be less than one right angle, that is, will be acute: therefore the angle $E HF$, which has been proved to be equal to it, will be an acute angle too. But it is an obtuse angle by supposition, which cannot be. Therefore the side AC of the triangle ABC , is not greater than the side DF of the triangle DEF .

After the same manner we demonstrate, that the side AC is not lesser than the side DF ; therefore the sides AC , DF are equal. In like manner also we prove, that these sides are equal, when the equal angles BAC , EDF , of the triangle ABC , DEF , are both obtuse. Therefore because these two triangles have three sides of the one equal to three sides of the other, the remaining angles of these two triangles will [by 4. 1.] be equal; that is, the angles ABC , DEF , and ACB , $D FE$.

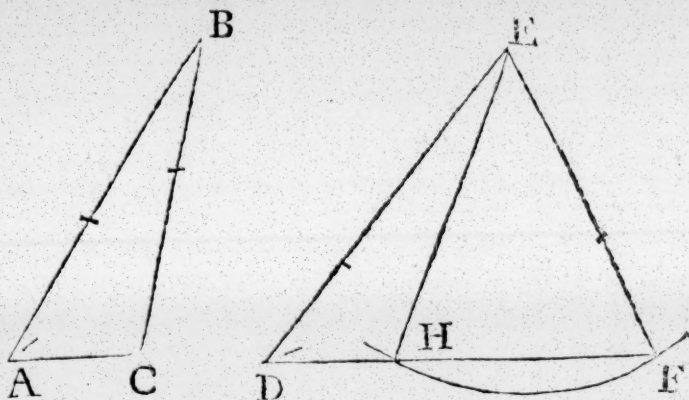
Therefore if two triangles have two sides of the one equal to two sides of the other, each to each, and one angle of the one equal to one angle of the other, but not that angle contained under the equal sides; and if the angles of each triangle opposite to one of the equal sides, be either both acute or both obtuse; I say, the remaining side of the one triangle will be equal to the remaining side of the other; and the remaining angles of the one equal to the remaining angles of the other. Which was to be demonstrated.

PROP. V. THEOR.

If two triangles have two sides of the one equal to two sides of the other, each to each, and one angle of the one equal to one angle of the other; but not that angle contained under the equal sides; and if one of the angles opposite to an equal side in one of the triangles be obtuse, and the other angle in the other triangle opposite to the correspondent equal side be acute; I say, the sum of these obtuse and acute angles will be equal to two right angles, and the difference of the angles contained under the equal sides, will be equal to the difference of the said acute and oblique angles.

Let

Let ABC , DEF be two triangles, having two sides AB , BC of the one equal to two sides DE , EF of the other, each to each; and one angle BAC of the one, equal to one



angle EDF of the other; but not that contained under the equal sides. Also let one of the angles, as ACB , opposite to an equal side AB , be obtuse; and the correspondent angle DFE , opposite to the other equal side DE , be acute: I say, the sum of the angles ACB , DFE will be equal to two right angles; and the difference of the angles ABC , DEF , contained as by the equal sides AB , BC , and DE , EF , will be equal to the difference of the obtuse and acute angles ACB , DFE , opposite to the equal sides AB , DE .

For about the angular point E , as a centre with the distance DF , describe a circle cutting the side DF of the triangle DEF in the point H , which will fall between the points D , F , because the angle DFE is supposed to be acute. Join EH .

Then because [by def. 15. 1.] the side EH is equal to the side EF , the angles EFH , EHF at the base, will [by 5. 1.] be equal. But since [by supposition] the side EF is equal to the side BC ; the sides HE , BC will be equal to one another: and since AB also is equal to DE , and the angle BAC equal to the angle EDF [by supposition]; therefore the angle ACB will be [by 4. of this] equal to the angle DHE , as well as the side AC equal to the side DH . And because [by 13. 1.] the angles DHE , EHF are equal to two right angles, and the angle DFE has been proved to be equal to the angle EHF , and the angle BCA to the angle DHE : therefore the angles ACB , DEF will be equal to two right angles.

Again, because all the angles of every triangle are [by 32. 1.] equal to two right angles; and the angle BAC [by

Let

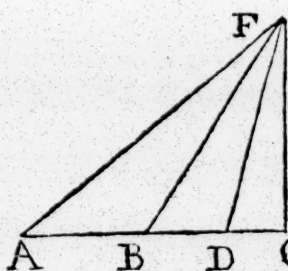
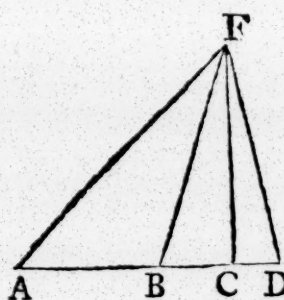
[by supposition] is equal to the angle EDF ; therefore will the sum of the angles ACB , ABC be equal to the sum of the angles DFE , DEF . Consequently, the difference of the angles DEF and ABC , will be equal to the difference of the angles ACB , EFD .

Therefore if two triangles have two sides of the one equal to two sides of the other, each to each, and one angle of the one equal to one angle of the other; but not that angle contained under the equal sides; and if one of the angles opposite to an equal side in one of the triangles be obtuse, and the other angle in the other triangle opposite to the correspondent equal side be acute; I say, the sum of these obtuse and acute angles will be equal to two right angles; and the difference of the angles contained under the equal sides, will be equal to the difference of the said acute and oblique angles. Which was to be demonstrated.

PROP. VI. THEOR.

In every triangle, the angle contained under the perpendicular drawn from the angle opposite to the base upon it, and the right line bisecting that angle, will be one half the difference of the angles at the base.

Let AFD be a triangle, whose perpendicular, drawn



from the angle F upon the base, is FC ; and let the right line FB bisect the angle AFD opposite to the base: I say,

the angle BFC contained under the perpendicular FC and the right line FB , will be equal to one half the difference of the angles FAD , FDA at the base.

For, first, let the perpendicular FC fall within the triangle. Then because [by 32. I.] the angles FAC , AFC , ACF are equal to two right angles, and the angle ACF [by supposition] is a right angle; the angles FAC , AFC will

will be one right angle; that is, the angles FAC , AFB , BFC will be one right angle (for AFC is equal to AFB , BFC). In like manner it is proved, that the angles CFD , CDF are equal to one right angle. Therefore the angles FAC , AFB , BFC are equal to the angles CFD , CDF . And if the angle BFC be added in common, the angles FAC , AFB , and twice the angle BFC , will be equal to the angles BFC , CFD , CDF ; that is, to the angles BFD , CDF . And if the equal angles AFB , BFD be taken away, there will remain the angle FAC , and twice the angle BFC , equal to the angle CDF . Therefore the angle CDF exceeds the angle FAC by twice the angle BFC ; and so the angle BFC is one half their difference.

Secondly, Let the perpendicular FC fall without the triangle AFD . Then because the external angle ADF of the triangle AFD is [by 32. I.] equal to the internal angles DFC , DCF , and the right angle DCF is equal to the angles FAC , AFC ; the angle ADF will be equal to the angles FAC , AFC , DFC ; that is, equal to the angles FAC , AFD , and twice the angle DFC ; or to the angles FAC , twice the angle BFD , and twice the angle DFC ; or equal to the angle FAC , and twice the angle BFC . Therefore the angle ADF exceeds the angle FAC , by twice the angle BFC ; and so the angle BFC will be one half the difference of them.

Wherefore in every triangle, the angle contained under the perpendicular drawn from the angle opposite to the base upon it, and the right line bisecting that angle, will be one half the difference of the angles at the base. Which was to be demonstrated.

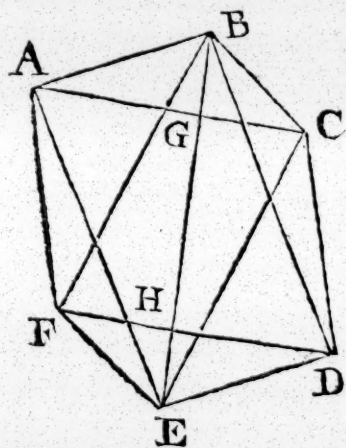
SCHOLIUM.

When the perpendicular FC falls without the triangle, it is evident, that the vertical angle AFD is the difference of the angles AFC , DFC at the perpendicular. But when the perpendicular FC falls within, the angle BFC itself (which is one half the difference of the angles at the base) is also one half the difference of the angles CFD , AFD at the perpendicular. For because the angles AFB , BFD are [by supposition] equal; if the angle BFC be added to both, the angle AFC will be equal to the angles BFD , BFC ; that is, to the angle CDF , and twice BFC : wherefore the angle BFC is one half the difference of the angles AFC , CDF .

PROP. VII. THEOR.

If there be any right-lined figure having six sides, and two contiguous sides of it be alternately equal and parallel to two other contiguous and opposite sides, each to each: I say, first, the two remaining sides will be equal and parallel; secondly, the opposite angles (viz. the first and fourth, second and fifth, third and sixth) will be equal to one another; thirdly, any diagonal joining two of those opposite angles, will divide the figure into two equal parts.

For let $ABCDEF$ be a right-lined figure having six sides; and let the two contiguous sides AB, BC be alternate equal and parallel to the two opposite contiguous sides FE, ED ; that is, FE equal and parallel to BC , and ED equal and parallel to AB : I say, first, that the remaining sides CD, AF will also be equal and parallel. Secondly, the opposite angles A, D ; B, E ; C, F ; will be respectively equal to one another. Thirdly, the diagonal BE will divide the figure into two equal parts.



For draw the diagonals AE, AC, BF, BD, CE, DF ; and let the diagonal BE cut the diagonals AC, DF in the points G, H .

Then because the sides AB, ED are equal and parallel; [by 33. 1.] the figure $ABDE$ will be a parallelogram, having its sides AE, BD equal and parallel, and the angles ABE, BED equal. In like manner, because the sides BC, EF are equal, the figure $FBCE$ will be a parallelogram; having the opposite sides FB, EC equal and parallel, and the angles FEB, CBE equal. Therefore the angles ABE, EBC will be equal to the angles BED, FEB ; that is, the whole angle ABC equal to the whole angle FED . Wherefore

fore since there are two triangles ABC , DEF , having two sides AB , BC of the one equal to two sides ED , FE of the other, each to each; and the angles ABC , FED contained under the equal sides are equal. [By 4. 1.] the base AC will be equal to the base DF , and the angle BAC equal to the angle FDE . But since the angle ABE has been proved to be equal to the angle BED , the outward angle BGC of the triangle ABG will be equal to the outward angle FHE of the triangle EHD . Therefore because the diagonal BE cuts the diagonals AC , FD , so that the alternate angles BGC , FHE are equal; [by 27. 1.] the diagonals AC , FD will be parallel. But they have been proved to be equal: wherefore [by 33. 1.] the sides AF , CD joining them will be equal and parallel.

Secondly, I say, the opposite angles A , D , or F , C , are equal. For because $ACDF$ has been proved to be a parallelogram, the opposite angles FAC , FDC ; AFD , ACD [by 34. 1.] will be equal. But since the angles CAB , ACB of the triangle ABC are respectively equal to the angles FDE , EDF of the triangle FED , the correspondent sides and angles of these triangles being equal to one another; therefore will the angles CAF , BAC be equal to the angles FDC , FDE ; that is, the angle FAB equal to the angle CDE . In like manner the angles AFD , EFD , as also the angles ACD , BCA will be equal; that is, the angle AFE equal to the angle BCD .

Thirdly, the diagonal BE divides the figure given into two equal parts.

For because [by 34. 1.] the diagonal BE of the parallelogram $ABDE$ divides that parallelogram into two equal triangles ABE , EDB ; and because the triangles AFE , ECD having each of the sides of the one equal to each of the sides of the other, are also equal. Therefore the triangles ABE , AFE , as also EDB , BCD , will be equal; that is, the trapezium $ABEF$ equal to the trapezium $BCDE$.

Therefore if there be any right lined figure having six sides, and two contiguous sides of it be alternately equal and parallel to the other contiguous and opposite sides, each to each: I say, first, the two remaining sides will be equal and parallel: Secondly, the opposite angles (*viz.* the first and fourth, second and fifth, third and sixth) will be equal to one another: Thirdly, any diagonal joining two of these

F

opposite

opposite angles, will divide the figure into two equal parts, which was to be demonstrated.

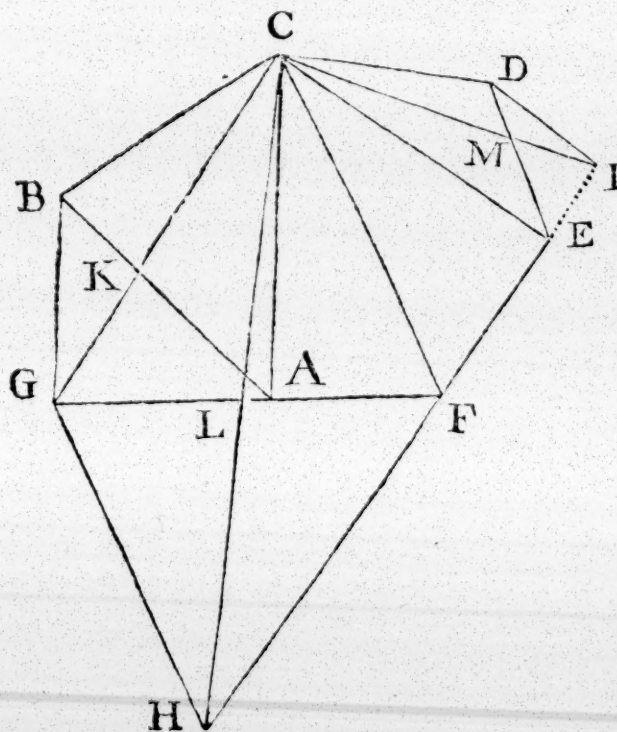
SCHOLIUM.

Hence a hexagon, whose opposite sides are all alternately parallel, may be called a parallelogram, that is, a figure having parallel sides; as well as the quadrilateral figure mentioned by Euclid at prop. 33. lib. 1. and, indeed, so may any other figure, having parallel sides, whatever be their number. Moreover, what Euclid has demonstrated at prop. 33, 34. lib. 1. of his four sided parallelograms, holds universally true of any parallelogram of six or more opposite sides, when these sides are alternately equal and parallel. Also, if the sides of any right-lined figure be alternately parallel, those that are opposite will be equal; and all the diagonals bisect one another in one and the same point. There are other uniform properties of these parallelograms, which I leave to others to discover and demonstrate.

PROP. VIII. PROBL.

To make a triangle equal to a given right-lined figure.

Let the given right-lined figure be $ABCDEF$: it is



required to make a triangle equal to the given right-lined figure $ABCDEF$.

From one angle c , draw the diagonal CA to the angle A next but one to the angle c . Continue out the side FA of the figure to G , being the first opposite side to the angle c ; and from the angle B , lying between

between the angles A and C , draw the right line BG [by 31. 1.] parallel to the diagonal CA , cutting the continuation of AF in the point G , and draw the right line CG , which let cut the side AB of the figure in the point K . Again, draw a second diagonal CF of the figure; and continuing out the next side FE of the figure to H , from the point G draw [by 31. 1.] the right line GH parallel to the second diagonal CF , meeting the continuation of the side EF of the figure in the point H , and draw the right line CH from the angle C of the figure. Let this meet the right line GF in the point L . Again, draw a third diagonal CE of the given figure, and continuing out the side FE of the figure to the point I , from the angle D draw the right line DI parallel to the diagonal CE , meeting the continuation of the side FE of the figure in the point I . And from the angle C draw the right line CI , meeting the side DE of the figure in the point M : I say, the triangle HCI will be equal to the given right-lined figure $ABCDEFGF$.

For because [by constr.] the right line BG is parallel to the diagonal AC of the figure, the triangles CBG , GAB standing upon the same base BG between these parallels, will [by 35. 1.] be equal to one another. And taking away from both the common triangle BGK , there will remain the triangle AKG equal to the triangle BKC . Therefore the right-lined figure $AKCDEF$, together with the triangle BKC , will be equal to the same right-lined figure $AKCDEF$, together with the triangle AKG . Wherefore the right-lined figure $G C D E F$ will be equal to the given right-lined figure $ABCDEFGF$.

Again, because [by constr.] the right line GH is parallel to the diagonal CF , the triangles GHC , GFH between them standing upon the base GH , will be equal [by 35. 1.] and taking away the common triangle GLH , the triangles HLF , GLC will be equal; and so the right lined figure $LFEDC$, together with the triangle HLF , that is, the right-lined figure $HEDC$, will be equal to the same right-lined figure $LFEDC$, together with the triangle GLC ; that is, equal to the right-lined figure $G C D E F$, or to the given right-lined figure $ABCDEFGF$; because these two figures have already been proved to be equal to one another. After the same manner we demonstrate, that the triangles DMC , EMI are equal to one another; and so

the triangle EMI , together with the right-lined figure $HEMC$, will be equal to the triangle CMD together with that figure; that is, the triangle HCI will be equal to the right-lined figure $HEDC$. But the right-lined figure $HEDC$ has been proved to be equal to the right-lined figure $GFEDC$; and this last right-lined figure, equal to the given right-lined figure $ABCDEF$. Therefore the triangle HCI will be equal to the given right-lined figure $ABCDEF$.

Therefore, &c. Which was to be demonstrated.

SCHOLIUM.

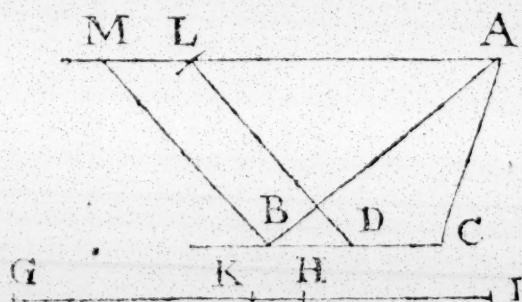
This is one of the most elegant and useful problems in right-lined Geometry. Some land measurers, who are used often to solve this problem upon paper, do it so easily and expeditiously, by means of a parallel ruler and compasses, as to obtain the triangle required without actually drawing any diagonals at all, or any parallels to them.

PROP. IX. PROBL.

To make a parallelogram equal to a given triangle whose four sides shall be equal to the three sides of the given triangle.

Let the given triangle be ABC : It is required to make a parallelogram equal to the given triangle ABC , having its four sides equal to the three sides of the given triangle.

Draw [by 31. 1.] the right line AM from the given point A parallel to the base BC of the triangle. And because, if neither of the angles B, C be a right angle [by 9. 1.] each of the sides AB, AC is greater than a perpendicular drawn from the angles A, B, C , or the point



to the opposite parallel. But if any of the angles be a right angle, then is, either of the sides perpendicular to the said parallels, both the sides AB, AC of the given triangle will be greater than twice the said perpendicular; and so half the sum of the sides will

greater than that perpendicular: that is, if the right line GH be taken equal to the side AB , and HI equal to the side AC , so that GI be the sum of the sides AB , AC , and GI be divided at K into two equal parts; this half GK will be greater than the perpendicular drawn from D to AM : therefore bisect [by 9. 1.] the base BC of the given triangle in the point D ; and about the centre D , with a distance equal to GK or KI , describe a circle cutting the parallel right line AM in some point, as L . Upon AM take LM equal to BD , and join the points B , M . Then [by 33. 1.] the figure $BMLD$ is a parallelogram: I say, the parallelogram $BMLD$ is equal to the given triangle ABC , and the sum of its sides BM , ML , LD , BD , will be equal to the sum of the sides AB , AC , BC of the given triangle.

For because [by constr.] the right line AM is parallel to the base BC of the triangle, and $BMLD$ is a parallelogram upon BD , half the base of the triangle, having its opposite side ML in AM , the parallelogram $BMLD$ will be [by 41. 1.] made equal to the given triangle ABC . Again, since [by constr.] DL , BM are each equal to GK , that is, to one half the sum of the sides AB , AC ; and accordingly, the sum of DL , BM is equal to the sum of the sides AB , AC of the triangle; and the right lines BD , ML are equal to the base BC of the triangle. Therefore the four sides DL , BM , BD , ML of the parallelogram $BMLD$ will be equal to the three sides AB , AC , BC of the given triangle: but the parallelogram $BMLD$ is also equal to the given triangle ABC .

Therefore, &c. Which was to be done.

PROP. X. THEOR.

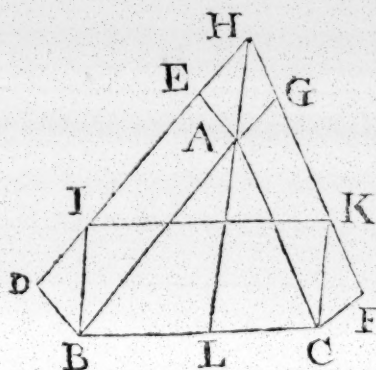
In every triangle any parallelograms described upon two sides are equal to the parallelogram described upon the remaining side, and whose other side is equal and parallel to the right line drawn from the angle contained under the first-mentioned sides, to that point wherein the sides of the parallelograms opposite to the sides of the triangle, when produced towards the angle, meet.

F 9

Let

Let there be any triangle ABC , and let any parallelograms $ABDE$, $ACFG$ be described upon the two sides AB , AC of it, whose two sides DE , FG , opposite to the assumed sides AB , AC of the triangle produced towards the angle A , contained under the said sides AB , AC , meet in the point H . Draw the right line AH : I say, the parallelograms AD , AF are equal to the parallelogram described upon the side BC , and its other side is equal and parallel to the right line AH .

For continue out the right line HA to cut the side BC of the triangle in the point L ; and from the points B , C draw [by 31. 1.] the right lines BI , CK parallel to the right line AH , and join the right line IK .



Then because [by supposition] $BIHA$, $CKHA$ are parallelograms; [by 34. 1.] BI , CK will each of them be equal to AH ; and so BI will be equal to CK ; and because [by 30. 1.] they are also parallel to one another, since each of them is parallel to the right line AH ; the right lines BC , IK will also be [by 33. 1.] parallel and equal. Therefore $BICK$ is a parallelogram upon BC , having its other side BI or CK equal and parallel to the right line AH ; which we are to prove to be equal to the parallelograms AD , AF . Therefore because the parallelograms AD , $ABIH$ are [by 35. 1.] equal; since they stand upon the same base AB , and are between the same parallels AB , HD ; and [by 35. 1.] the parallelogram $ABIH$ is equal to the parallelogram IL , because they have both the same base BI , and stand between the same parallels BI , LH ; the parallelogram AD will be equal to the parallelogram IL . After the same manner we demonstrate, that the parallelogram AF is equal to the parallelogram KL : therefore the parallelograms AD , AF will be equal to the parallelogram BK .

Therefore in every triangle, any parallelograms described upon two sides are equal to the parallelogram described upon the remaining side, and whose other side is equal and parallel to the right line drawn from the angle contained under

der the first-mentioned sides, to that point wherein the sides of the parallelograms opposite to the sides of the triangle, when produced towards that angle, meet. Which was to be demonstrated.

SCHOLIUM.

This theorem is to be found in Pappus's mathematical collections. It is much more general than the 47th proposition of the first book of Euclid, which is only that particular case of this, where the parallelograms described upon the two assumed sides of the triangle are each of them squares, and the angle of the triangle contained under these two sides, is a right angle.

PROP. XI. PROBL.

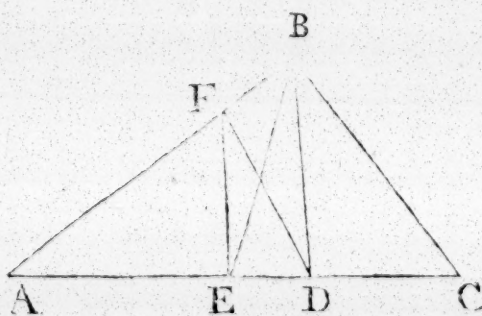
To divide a given triangle into two equal parts, by a right line drawn from any given point in one of its sides.

Let there be a given triangle ABC , and D any given point in one of its sides: It is required to draw a right line from the given point D that shall divide the triangle ABC into two equal parts.

Bisect [by 10. 1.] the side AC of the triangle in the point E , and join the right line EB ; then if the point E falls in the point D , the thing is done; for [by 38. 1.] the triangles ABE , BCE are equal; and each of them will be one half of the given triangle ABC . Also, if the given point be in either of the angles of the triangle, the thing is done by drawing a right line from that angle to the middle point of the opposite side.

In all other cases, draw the right line EF [by 31. 1.] parallel to the right line DB , cutting the side AB of the triangle in the point F , and join the right line DF .

Then because [by constr.] EF is parallel to the right line BD , the triangles FED , FBE standing upon



upon the same base FE , between those parallels, will [by 35. 1.] be equal to one another; and adding to both, the common triangle AFE ; the triangles AFE , FBE will be equal to the triangles AFE , EFD ; that is, the triangles ABE , AFD will be equal to one another. But it has been proved, that the triangle ABE is one half the given triangle ABC : therefore also the triangle AFD will be one half of the given triangle ABC ; and so the remaining trapezium $FBCD$ will be one half of the given triangle ABC . Wherefore the right line DF , drawn from the given point D in the side of the given triangle ABC , will divide it into two equal parts: which was to be done.

PROP. XII. PROBL.

To divide a given trapezium into two equal parts by a right line drawn from a given point in the middle of one of its sides.

Let $ABCD$ be a given trapezium, and E a given point in the middle of one of its sides: it is required to draw a right line from the given point E , that will divide the given trapezium into two equal parts.

For [by 31. 1.] draw the right line FC from the angle C of the figure parallel to the side AD , cutting the side AB in the point F ; bisect [by 10. 1.] the right line FC in the point G . And having drawn a right line from the given point E to the angle B of the figure, draw a right line GH from G parallel to the right line EB , meeting the side BC of the figure in the point H ; and join the right line EH ; and the business is done.

For join the right lines EF , EC , EG , BG .

Then because AE is equal to ED , and FC is parallel to AD [by constr.]; the triangles AFE , ECD will be equal to one another [by 37. 1.]. And because FG , GC are equal [by constr.]; the triangles FEG , GEC will [by 37. 1.] be equal to one another; as also the triangles FEG , GEC . Therefore the triangles AFE , FEG will be equal.

to the triangles $E C D$, $E G C$; that is, the trapezium $A F G E$ will be equal to the trapezium $E G C D$. And since the triangles $F B G$, $B G C$ are equal, the trapezium $A B G E$ will be equal to the figure $E G B C D$: consequently, the trapezium $A B G E$ will be one half the given trapezium $A B C D$. Again, because [by constr.] $G H$ is parallel to $E B$, the triangles $E G B$, $E B H$ will [by 37. 1.] be equal to one another. Therefore the triangles $A B E$, $E B H$, will be equal to the triangles $A B E$, $E G B$. But the trapezium $A B G E$ consists of the triangles $A B E$, $E B G$, and the trapezium $A B H E$ of the triangles $A B E$, $E B H$: wherefore the trapeziums $A B H E$, $A B G E$ will be equal to one another. But the trapezium $A B G E$ has been proved to be equal to one half of the given trapezium $A B C D$. Therefore the trapezium $A B H E$ will also be equal to one half of the given trapezium $A B C D$.

Therefore a given trapezium is divided into two equal parts, by a right line drawn from a given point in the middle of its side. Which was to be done.

The problem is almost as easily resolved, when the given point is not in the middle of one of the sides; for it is only drawing a right line from such a point joining the point H , found as above, and through the middle point E drawing a right line parallel to this line. Then a right line drawn from the given point to that wherein this parallel intersects the opposite side $B C$ will divide the trapezium into two equal parts.

E U C L I D's E L E M E N T S, B O O K II.

D E F I N I T I O N S.

1. **E**VERY right-angled parallelogram is said to be contained under the two right lines, comprehending a right angle ^a.

2. In every parallelogram, any one of the parallelograms that are about a diameter, with the two complements, is called a gnomon.

^a Here Euclid, for brevity's sake, says, *Every right-angled parallelogram is contained under two right lines forming a right angle*; though, indeed, every parallelogram is comprehended or contained under four right lines. But as the opposite sides of every parallelogram are equal to one another, any of the two sides which contain a right angle, may not unaptly be said to contain the whole parallelogram.

P R O P O S I T I O N. I. T H E O R E M.

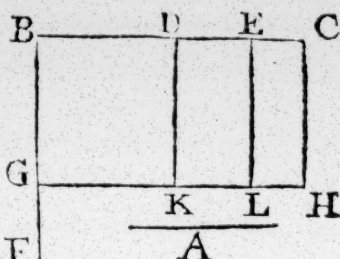
If there be two right lines, and one of them is cut or divided into any parts whatsoever. The rectangle contained under the two right lines is equal to all the rectangles contained under the undivided line, and the several segments or parts of the other line.

LET there be two right lines, A and BC, and let BC be any how cut or divided in the points D, E: I say, the rectangle contained under the right lines A, BC, is equal

to

to the rectangle which is contained under the right lines A, BD , and to the rectangle contained under A, DE , and to that which is contained under A, EC .

For [by prop. II. B. I.] from the point B draw the right line BF at right angles to BC , and make BG equal to A . And thro' G [by 31. I.] draw GH parallel to BC . And through the points D, E, C draw the right lines DK, EL, CH parallel to BG .



Then the rectangle BH is equal to the rectangles BK, DL, EH . And BH is the rectangle under A, BC ; for it is contained under GB, BC , and BG is equal to A : But the rectangle BK is contained under A, BD ; for it is contained under GB, BD , and GB is equal to A : and the rectangle DL is contained under the right lines A, DE ; because DK , that is, BG , is equal to A : And in like manner, the rectangle EH , is contained under the right lines A, EC : Therefore the rectangle under the right lines A, BC , is equal to all the rectangles under A, BD, DE, EC , and also under A and EC .

Therefore if a right line be cut or divided into any parts soever, the rectangle contained under the two right lines, is equal to all the rectangles contained under the undivided line, and the several segments or parts of the divided line. Which was to be demonstrated.

PROP. II. THEOR.

*If a right line be any how divided into two parts, the rectangles contained under the whole line and each of the parts, are equal to the * square of the whole line^b.*

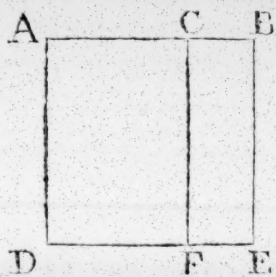
For let the right line AB be any how divided into two parts at C : I say, the rectangle contained under the right lines AB, BC , together with the rectangle contained under BA, AC , are equal to the square of the right line AB .

* For brevity's sake, we call the square described upon a line, the square of a line.

For

For [by 46. I.] describe the square $ADEB$ of AB ; and through C [by 31. I.] draw CF parallel to AD or BE .

Then is AE equal to both the rectangles AF , CE : But



AE is the square of the right line AB , and the rectangle AF is contained under the right lines BA , AC ; for it is contained under the right lines DA , AC , whereof AD is equal to AB ; and the rectangle CE is contained under the right lines AB , BC ; for the right line BE is equal to the right line AB :

Therefore the rectangle contained under the right lines AB , AC , together with the rectangle contained under AB , BC , is equal to the square of AB .

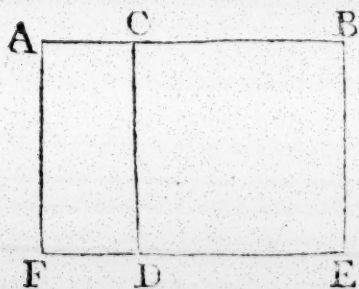
If therefore a right line be any how divided into two parts, the rectangles contained under the whole line and each of the parts, will be equal to the square of the whole line. Which was to be demonstrated.

^b Although Euclid proposes this second theorem of a right line divided any how only into two parts; the same thing may be demonstrated after the same manner, if a right line be divided into any number of parts.

PROP. III. THEOR.

If a right line be divided into any two parts, the rectangle contained under the whole line, and one of the parts, is equal to the rectangle contained under the parts, and the square of the part aforesaid.

For let the right line AB be divided in the point C into any two parts: I say, the rectangle under AB , BC is equal to the rectangle under AC , CB , together with the square of the right line BC .



For [by 46. I.] describe the square $CDEB$ of BC , produce ED to F , and through A draw [by 31. I.] AF parallel to CD or BE .

Then

Then the rectangle AE will be equal to the rectangles AD , CE , and AE is the rectangle contained under the right lines AB , BC ; for it is contained under the right lines AB , BE , whereof BE is equal to BC : But the rectangle AD is contained under AC , CB , since DC is equal to CB , and DB is the square of BC : Therefore the rectangle under AB , BC is equal to the rectangle under AC , CB , together with the square of BC .

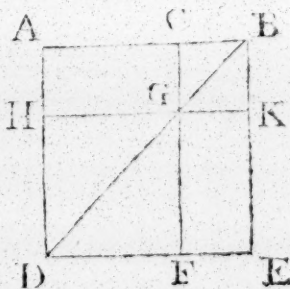
If therefore a right line be divided into any two parts, the rectangle contained under the whole line and one of the parts, is equal to the rectangle contained under the parts, and the square of the part aforesaid. Which was to be demonstrated.

PROP. IV. THEOR.

If a right line be divided into any two parts, the square of the whole line, is equal to both the squares of the parts, together with twice the rectangle contained under the parts.

For let the right line AB be divided by the point C into any two parts: I say, the square of AB is equal to both the squares of AC , CB , together with twice the rectangle contained under the parts AC , CB .

For [by 46. I.] describe the square $ADEB$ of AB . Join BD , and through C draw [by 31. I.] the right line CGF parallel to AD or BE ; but through G , the right line HK parallel to AB , or DE .



Then because CF is parallel to AD , and the right line BD falls upon them, the outward angle BGC will be [by 29. I.] equal to the inward opposite angle ADB . But [by 5. I.] the angle ADB is equal to the angle ABD , because the side BA is equal to the side AD ; and so the angle CGB is equal to the angle GBE : therefore the side BC [by 6. I.] is equal to the side CG . But likewise CB is [by 34. I.] equal to the side GK , and CG to EK . Therefore the side GK is equal to KB , and the parallelogram $CGKB$ is equilateral: I say, it is right-angled too. For because CG is parallel to EK , and the

right line CB falls upon them: the angles KBC , GCB are [by 29. 1.] equal to two right angles. But [by 30. def. 1.] the angle KBC is a right angle, and so the angle GCB is a right angle too: wherefore [by 34. 1.] the opposite angles CGK , GKB will be right angles. Therefore $CGKB$ is a rectangle. But it has been proved to be equilateral: wherefore $CGKB$ is a square, described upon BC . By the same reason also the parallelogram HF is a square, *viz.* the square of HG , that is, of AC : therefore HF , CK are the squares of AC , CB . And because the rectangle AG [by 43. 1.] is equal to the rectangle GE , and AG is contained under the right lines AC , CB ; for GC is equal to CB ; therefore GE is equal to the rectangle contained under AC , CB : Wherefore the rectangles AG , GE are equal to twice the rectangle under the right lines AC , CB . But HF , CK are the squares of AC , CB . Therefore the four parallelograms HF , CK , AG , GE are equal to the squares of AC , CB , together with twice the rectangle under the right lines AC , BC . But HF , CK , AG , GE make up the whole $ADEB$, *viz.* the square of AB . Wherefore the square of AB is equal to both the squares of AC , BC together, with twice the rectangle contained under AC and CB .

Therefore if a right line be divided into any two parts, the square of the whole line is equal to both the squares of the parts, together with twice the rectangle contained under the parts. Which was to be demonstrated.

Otherwise:

I say, the square of AB is equal to both the squares of AC , CB , together with twice the rectangle under AC , CB .

For because, in the same figure, BA is equal to AD , the angle ABD [by 5. 1.] will be equal to the angle ADB ; and since [by 32. 1.] the three angles of every triangle, all together, are equal to two right angles. But the angle BAD is a right angle. Therefore the remaining angles ABD , ADB are both together equal to one right angle. But they are equal to one another: Wherefore the angles ABD , ADB are each one half a right angle. But BCG is a right angle, for it is equal to the inward opposite angle at A ; and so the remaining angle CGB is one half a right angle: accordingly, the angle CGB is equal to the angle

CBG ,

CBG , and the side BC [by 6. 1.] equal to the side CG . But CB [by 34. 1.] is equal to KG , and CG to BK : wherefore CK is equilateral. But it has one right angle CBK ; therefore it is a square, *viz.* that of CB . For the same reason HF is a square too, being equal to the square of AC : Therefore the squares CK , HF are equal to the squares of AC , CB . Again, because [by 43. 1.] the rectangle AG is equal to EG , but AG is the rectangle contained under the right lines AC , CB , for CG is equal to CB ; the rectangle EG is equal to that under the right lines AC , CB : Wherefore AG , GE are equal to twice the rectangle under the right lines AC , CB ; but also CK , HF , AG , GE are equal to the squares of AC , CB , together with twice the rectangle under AC , CB . But CK , HF and AG , GE , make up the whole square of AB .

Therefore the square of AB is equal to both the squares of AC , CB , together with twice the rectangle contained under AC , CB . Which was to be demonstrated.

Corol. From hence it is manifest, that the parallelograms in squares that are about the diameter, are squares themselves.

PROP. V. THEOR.

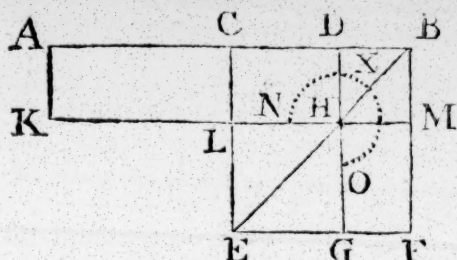
If a right line be divided into two equal, and into two unequal parts, the rectangle contained under the unequal parts of the whole, together with the square of the right line between the two points of division, will be equal to the square of half the line^c.

For let any right line AB be divided at the point C into equal parts, and into unequal ones at the point D . I say, the rectangle contained under the right lines AD , DB , together with the square of CD , is equal to the square of the half line CB .

For [by 46. 1.] describe the square $CEFB$ of BC ; join BE ; through D draw [by 31. 1.] DHG parallel to CE or BF ; through H draw KLM parallel to CB or EF ; and through A draw AK parallel to CL or BM .

Now

Now because [by 43. I.] the complement CH is equal to the complement HF , add DM , which is common to



E G I C H, which is common to

both. And then the whole AH will be equal to DF , DL . But AH is the rectangle under AD , DB , for DH is equal to DB ; but FD , DL is the gnomon NXO : Therefore the gnomon NXO is equal to the rectangle contained under the right lines AD , DB . Add LG , which [by Cor. 4. 2.] is equal to the square of CD , to both, and then the gnomon NXO and LG are equal to the rectangle under the right lines AD , DB , together with the square of CD . But the gnomon NXO and LG , make up the whole square $CEFB$, viz. that of CB : Therefore the rectangle AD , DB , together with the square of CD , is equal to the square of CB .

If therefore a right line be divided into two equal, and into two unequal parts, the rectangle contained under the unequal parts, together with the square of the right line between the two points of division, will be equal to the square of half the line. Which was to be demonstrated.

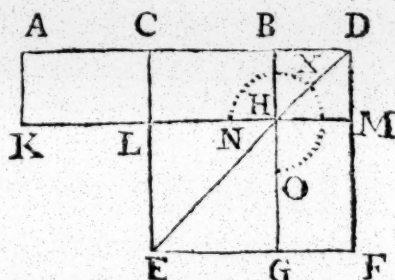
' This theorem may be demonstrated otherwise thus : Because
 the square of CB is [by 4. 2.] equal to the squares of CD , DB ,
 A C D B together with twice the rectangle
 under DB , CD , and the rectangle under
 DB , CB [by 3. 2.] equal to the rectangle under DB , CD ,
 together with the square of DB ; the square of CB will be equal
 to the remaining square of CD , together with the remaining
 rectangle under DB , CD , and the rectangle under DB , CB , or
 under DB , AC . But the rectangle under DB and the whole AB ,
 is equal to the rectangles under DB , AC , and under DB , CD
 [by 1. 2.]. Therefore the square of CB will be equal to the
 square of CD , together with the rectangle under DB , AD ; that
 is, the rectangle under AD , DB , together with the square of
 CD , will be equal to the square of CB . Which was to be de-
 monstrated.

PROP. VI. THEOR.

If a right line be divided into two equal parts, and any other right line be joined to it in the same direction; the rectangle contained under the line, composed of the whole line and the joined line, and the joined line, together with the square of the half line, is equal to the square of the line composed of the half and the joined line, taken as one line^d.

For let any right line AB be bisected in the point C , and any other right line BD be joined to it in the same direction. I say, the rectangle contained under AD , DB together with the square of CB , is equal to the square of CD .

For [by 46. I.] describe the square $CEFD$ of CD ; join DE ; thro' the point B [by 31. I.] draw BHG parallel to CE or DF ; through the point H , draw KLM parallel to AD or EF ; and through A draw AK parallel to CL or DM .



Now because AC is equal to CB , the rectangle AL will be [by 36. I.] equal to the rectangle CH . But [by 43. I.] CH is equal to HF ; and therefore will AL be equal to HF . Add CM , which is common to both; then the whole AM is equal to the gnomon NXO . But AM is the rectangle contained under the right lines AD , DB ; for DM [by cor. 4. 2.] is equal to DB : therefore the gnomon NXO is equal to the rectangle contained under the right lines AD , DB . Again, add LG , which is common to both, being equal to the square of CB ; then the rectangle under AD , DB , together with the square of BC , is equal to the gnomon NXO and to LG . But the gnomon NXO and LG make up the whole square $CEFD$, that is, the square of CD : wherefore the rectangle under AD , DB , together with the square of BC , is equal to the square of CD .

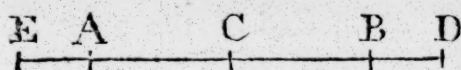
If therefore a right line be divided into two equal parts, and any right line be joined to it in the same direction; the rectangle contained under the whole line and the joined

G

line,

line, and the joined line, together with the square of the half line, is equal to the square of the right line composed of the half and the joined line, taken as one line. Which was to be demonstrated.

^d This theorem may be otherwise demonstrated thus: Let the right line AB be bisected in the point C , and let any right line BD be added to it, having the same direction. I say, the rectangle under the whole line AD , and the added line BD , together with the square of the half line CB , will be equal to the square of the right line CD . For let the right line AE be added to AB on the contrary side equal to BD ; then the whole line ED will be cut equally in



C , and unequally in B ; since EA , AC are equal to DB , BC . The right line EB also will be equal to the right line AD , for EA is equal to DB and AB is common to both. Therefore [by 5.2.] the rectangle under the unequal parts EB , BD , together with the square of the intermediate segment CB ; that is, the rectangle under AD , BD , together with the square of the half line CB , will be equal to the square of the half line CD ; that is, to the square of the right line compounded of half the given line AB , together with the added line BD , taken as one line. Which was to be demonstrated.

PROP. VII. THEOR.

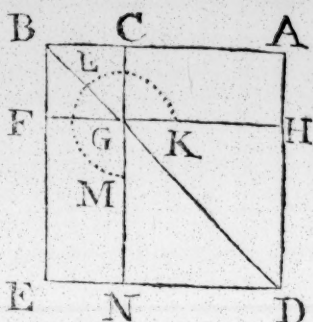
If a right line be divided into any two parts, the square of the whole line, together with that of one of the parts, will be equal to twice the rectangle contained under the whole line and the said part, together with the square of the remaining part^e.

For let any right line AB be divided by the point C into any two parts. I say, both the squares of AB , BC are equal to twice the rectangle contained under the right lines AB , BC , together with the square of AC .

For [by 46. 1.] describe the square $ADEB$ of AB , and construct [or make] the figure.

Now because [by 43. 1.] the rectangle AG is equal to the rectangle GE , add CF , which is common, to both; then the whole AF is equal to the whole CE : therefore the

the rectangles AF , CE are double to the rectangle AF . But AF , CE are the gnomon KLM and the square CF : wherefore the gnomon KLM and the square CF will be double to the rectangle AF . But twice the rectangle under AB , BC is double to AF : for [by cor. 4. 2.] BF is equal to BC . Therefore the gnomon KLM , and the square CF , are equal to twice the rectangle under the right lines AB , BC . Add HN , which is common, to both, *viz.* the square of AC ; then the gnomon KLM and the squares CF , HN , are equal to twice the rectangle under the right lines AB , BC , together with the square of AC . But the gnomon KLM and squares CF , HN , make up the whole $ADEB$, together with CF ; which are the squares of AB , BC : Therefore the squares of AB , BC are equal to twice the rectangle under AB , BC , together with the square of AC .



Therefore if a right line be divided into any two parts, the square of the whole line, together with the square of one of the parts, will be equal to twice the rectangle contained under the whole line and the said part, together with the square of the remaining part. Which was to be demonstrated.

* This theorem may be demonstrated otherwise thus: Because the square of AB is [by 4. 2.] equal to the squares of AC , CB , together with twice the rectangle under AC , CB , if the common square of AC be added, the squares of AB , AC will be equal to twice the square of AC , and AC to the square of CB , together with twice the rectangle under AC , CB . But [by 3. 2.] the rectangle under AB , AC , is equal the rectangle under AC , CB , together with the square of AC : wherefore twice the rectangle under AB , AC , will be equal to twice the rectangle under AC , CB , together with twice the square of AC . Therefore the squares of AB , AC , are equal to the remaining square of CB , together with twice the rectangle under AB , AC . Which was to be demonstrated.

PROP.

PROP. VIII. THEOR.

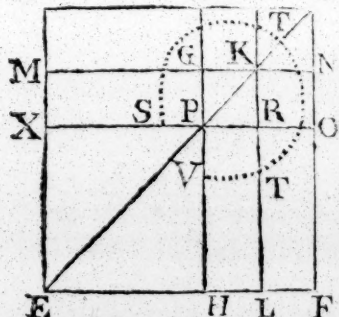
If a right line be divided into any two parts, four times the rectangle contained under the whole line and one of the parts, together with the square of the remaining part, is equal to the square of the whole line and the aforesaid part, taken both together as one line^f.

For let the right line AB be divided into any two parts at C : I say, four times the rectangle under AB , BC , together with the square of AC , is equal to the square of AB , BC taken as one line.

For continue out the right line AB to D ; make BD equal to CB . Describe [by 46. I.] the square $A E F D$ of AD , and construct the double figure.

Then because CB is equal to BD , but CB [by 34. I.] is equal to GK , and BD to KN ; therefore GK will be equal to KN . And for the same reason also PR will be equal to RO .

And because CB is equal to BD , and GK to KN ; the rectangle CK [by 36. I.] will be



equal to the rectangle BN , and the rectangle GR to RN . But [by 43. I.] CK is equal to RN , for they are the complements of the parallelogram CO . Therefore BN is equal to GR , and the four rectangles BN , KC , GR , RN are equal to one another, and so are four times the rectangle CK . Again, because CB is equal to BD , and BD to BK ; that is [by 34. I.] equal to CG ; but CB is equal to GK , that is, to GP : Therefore CG is equal to GP . But PR is equal to RO . Wherefore the rectangle AG [by 36. I.] will be equal to the rectangle MP , and the rectangle PL will be equal to RF . But MP [by 43. I.] is equal to PL ; for they are the complements of the parallelogram ML . Wherefore also AG is equal to RF . Wherefore the four rectangles AG , MP , PL , RF , are equal to one another; and so are four times AG . But it has been proved, that the four rectangles CK , BN , GR , RN , are four times CK : wherefore the eight rectangles which

which make up the gnomon STV are four times the rectangle AK . And because AK is the rectangle under AB , BD , for BK [by cor. 4. 2.] is equal to BD , four times the rectangle contained under AB , BD will be four times the rectangle AK . But it has been proved, that the gnomon STV is four times AK : wherefore four times the rectangle under AB , BD is equal to the gnomon STV . Let HX , which is [by cor. 4. 2.] equal to the square of AC , be added to both; then four times the rectangle contained under AB , BD , together with the square of AC , will be equal to the gnomon STV and the square HX . But the gnomon STV and HX , make up the whole square $AEDF$ described upon the right line AD : therefore four times the rectangle contained under the right lines AB , BC , together with the square of AC , is equal to the square of AD , that is to the square of AB , BC taken as one line.

Therefore if a right line be divided into any two parts, four times the rectangle contained under the whole line and one of the parts, together with the square of the remaining part, is equal to the square of the whole line and the aforefaid part, taken both together as one line. Which was to be demonstrated.

[This theorem may be demonstrated otherwise thus: Because the square of AD is [by 4. 2.] equal to the squares of AB , BD , together with twice the rectangle under AB , BD ; that is, to the squares of AB , BC , together with A C B D twice the rectangle under AB , BC . But the squares of AB , BC are [by 7. 2.] equal to twice the rectangle under AB , BC , together with the square of AC : therefore the square of AD will be equal to four times the rectangle under AB , CB , together with the square of AC . Which was to be demonstrated.]

PROP. IX. THEOR.

If a right line be divided into two equal and into two unequal parts, the squares of the unequal parts are double to the squares of half the line, and of the right line between the points of division.

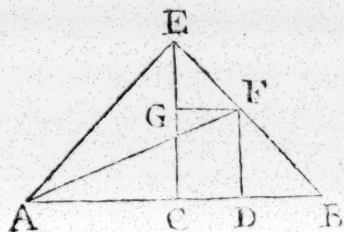
For let any right line AB be divided into equal parts at C , and into unequal ones at D : I say, the squares of AD , DB are double to the squares of AC , CD .

G 3

For

For [by 11. 1.] draw CE from the point C at right angles to AB , make it equal to AC , or CB ; join EA , EB ; through D [by 31. 1.] draw DF parallel EC ; but through F draw FG parallel to AB , and join AF .

Then because AC is equal to CE , the angle EAC [by 5. 1.] will be equal to the angle AEC . And since the angle at C is a right angle, the remaining angles AEC , EAC [by 32. 1.] will be equal both to one right angle. These will therefore be equal to one another; and so each of them one half a right angle. By the same reason the angles CEB , ECB are each one half a right angle: wherefore the whole angle AEB is a right angle. And because the angle GEF is one half a right angle; and ECF is a right angle; for [by 29. 1.] it is equal to the inward opposite



angle ECB ; the remaining angle ECF will be one half a right angle: therefore the angle GEF is equal to the angle ECF ; and [by 6. 1.] the side EG is equal to the side CF . Again, because the angle at B is one half a right angle, and FDB is a right angle; since [by 29. 1.] it is equal to the inward opposite angle ECB ; the remaining angle BFD will be one half a right angle: therefore the angle EBF is equal to the angle BFD : and so the side DF [by 6. 1.] is equal to the side DB . And because AC is equal to CE , the square of AC will be equal to the square of CE : therefore the squares of AC , CE are double to the square of AC . But [by 47. 1.] the square of EA is equal to the squares of AC , CE , for the angle ACE is a right angle: therefore the square of EA is double to the square of AC . Again, because EG is equal to GF , and the square of EG is equal to the square of GF ; the squares of EG , GF are double to the square of GF . But [by 47. 1.] the square of EF is equal to the square of EG , GF : therefore the square of EF will be double to the square of GF . But [by 34. 1.] FG is equal to CD : therefore the square of EF will be double to the square of CD . But also the square of AE is double to the square of AC ; therefore the squares of AE , EF are double to the squares of AC , CD : but [by 47. 1.] the square of AF is equal to the squares of AE , EF , because AEF is a right angle: therefore the square of AF

double

double to the squares of AC , CD . But [by 47. 1.] the squares of AD , DF are equal to the square of AF , for the angle at D is a right angle: wherefore the squares of AD , DF are double to the squares of AC , CD . But DF is equal to DB ; therefore the squares of AD , DB will be double to the squares of AC , CD .

If therefore a right line be divided into two equal and into two unequal parts, the squares of the unequal parts are double to the squares of half the line, and of the right line between the points of division. Which was to be demonstrated.

§ This theorem may be demonstrated otherwise thus: Because the square of the right line AD [by 4. 2.] is equal to both the squares of AC , CD , together with twice the rectangle under AC , CD . If the common square of BD be added, the two squares of AD , DB will be equal to the $A \quad C \quad D \quad B$ three squares of AC , CD , DB , together with twice the rectangle under AC , CD , or under BC , CD ; but the squares of BC , or AC , and that of CD , are [by 7. 2.] equal to the square of DB , together with twice the rectangle under BC , CD : therefore the squares of AD , DB are equal to twice the squares of AC , CD ; and so the squares of AD , DB are double to the squares of AC , CD . Which was to be demonstrated.

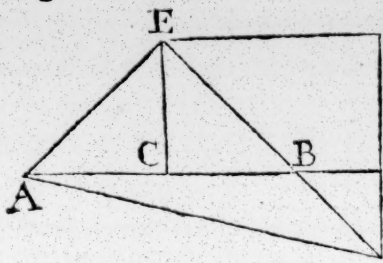
PROP. X. THEOR.

If a right line be divided into two equal parts, and any right line be joined to it in the same direction, the square made of the whole and joined line, and the square of the joined line taken together, are double to the square of one half the line, and to the square of the line made of one half the line and the joined line, taken as one line^h.

For let the right line AB be divided at C into two equal parts, and let any right line BD be joined to it in the same direction: I say, the squares of AD , DB are double to the squares of AC , CD .

For draw from C [by 11. 1.] the right line CE at right angles to AB ; make it equal to AC or CB ; join AE , EB ; through E [by 31. 1.] draw EF parallel to AD ; and through D draw DF parallel to CE . And because the right line EF falls upon the two parallels EC , FD , the angles CEF , EFD are [by 29. 1.] equal to two right angles:

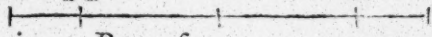
angle: therefore the angles FEB , EFD will be less than two right angles. But right lines infinitely produced from angles less than two right angles [by ax. 11.] will meet one another: Therefore EB , FD produced towards B , D , G will meet. Let them be produced, and meet in the point G : join AG .



Now because AC is equal to CE , the angle AEC [by 5. 1.] will be equal to the angle EAC . But the angle at C is a right angle: wherefore EAC , AEC are [by 32. 1.] each half a right angle. By the same reason, CEB , EBC are each one half a right angle: therefore AEB is a right angle. And because EBC is one half a right angle, the angle DBG [by 15. 1.] will be one half a right angle. But [by 29. 1.] BDG is a right angle too; for it is equal to the alternate angle DCE : wherefore the remaining angle DGB is one half a right angle; and so the angle DGB is equal to the angle DBG : wherefore the side BD will be [by 6. 1.] equal to the side GD . Again, because EGF is one half a right angle, and the angle at F is a right angle, for [by 34. 1.] it is equal to the opposite angle at C ; the remaining angle FEG will be one half a right angle: wherefore the angle EGF is equal to the angle FEG ; and so the side GF [by 6. 1.] is equal to the side EF . And since EC is equal to CA , and the square of EC equal to the square of CA ; the squares of EC , CA are double to the square of EA . But the square of EA [by 47. 1.] is equal to the squares of EC , CA : therefore the square of EA is double to the square of AC . Again, because GF is equal to EF , and the square of GF is equal to the square of EF ; the squares of GF , EF are double to the square of EF . But [by 47. 1.] the square of EG is equal to the squares of GF , EF ; therefore the square of EG will be double to the square of EF : but EF is equal to CD : wherefore the square of EG will be double to the square of CD . But it has been demonstrated, that the square of EA is double to the square of AC : therefore the squares of AE , GE are double to the squares of AC , CD . But the square of AG [by 47. 1.] is equal to the squares of AE , EG ; therefore the square of AG is double to the squares of AC , CD . But

[by 47. 1.] the square of AG is equal to the squares of AD , DG : therefore the squares of AD , DG are double to the squares of AC , CD . But DG is equal to DB . Wherefore the squares of AD , DB are double to the squares of AC , CD .

If therefore a right line be divided into two equal parts, and any right line be joined to it in the same direction, the square of the whole and joined line, together with the square of the joined line, are equal to twice the square of one half the line, and the square of the line made of the half line, and the joined line taken as one line. Which was to be demonstrated.

^h This theorem may be demonstrated otherwise thus: Let the right line AB be bisected in C , and let any right line BD be added to it in the same direction. I say, the squares of AD , BD together, are double to the squares of AC , CD together; for having added the right line AE to AB on the contrary side thereof (*viz.* towards A) equal to BD ; E A C B D
then the whole ED will be 
cut equally in C and unequally in B . Because EA , AC are equal to DB , BC , the right line EB will also be equal to the right line AD ; since EA is equal to DB , and AB common to both. Therefore [by 9. 2.] the squares of EB , BD , that is, of AD , BD , will be double to the squares of EC , CB , that is, of CD , AC . Which was to be demonstrated.

PROP. XI. PROBL.

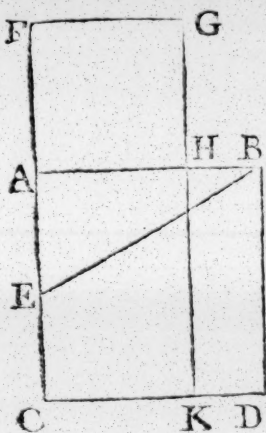
To divide a right line into two such parts, that the rectangle contained under the whole line and one of the parts, shall be equal to the square of the other partⁱ.

Let AB be a given right line: it is required to divide it into two such parts, that the rectangle contained under the whole line and one of those parts, shall be equal to the square of the other part.

For [by 46. 1.] describe the square $ABDC$ of AB , and [by 10. 1.] divide AC into two equal parts at E ; join BE ; produce CA to F ; and [by 3. 1.] make EF equal to BE ; also describe FH the square of AF , and produce GH to K : I say, AB is so divided in H , that the rectangle under AB , BH is equal to the square of AH .

For

For because the right line AC is divided into two equal parts at E , and AF is joined to it in the same direction; the rectangle under CF , FA , together with the square of AE [by 6. 2.] will be equal to the square of EF . But EF is equal to EB : wherefore the rectangle under CF , FA , together with the square of AE , is equal to the square of EB . But [by 47. 1.] the squares of BA , AE are equal to the square of EB ; for the angle at A is a right angle: therefore the rectangle under CF , FA , together with the square of AE , is equal to the squares of BA , AE . Take



away from both the common square of AE ; and then there will remain the rectangle under CF , FA equal to the square of AB . But the rectangle KF is the rectangle under CF , FA ; for AF is equal to FG , and AD is the square of AB : therefore the rectangle FK is equal to the square AD . Take away the common rectangle AK ; and there will remain FH equal to DH . But DH is the rectangle under AB , BH , since AB is equal to BD , and FH is the square of AH : therefore the rectangle under AB , BH is equal to the square of AH .

Wherefore the given right line AB is so divided in the point H , that the rectangle under AB , BH is equal to the square of AH . Which was to be demonstrated.

ⁱ Numbers are applicable to all the other propositions of this book but this, there being no number that can be divided into two others such, that the product of the whole by one of the parts, shall be equal to the square of the other part.

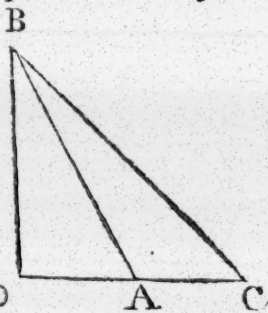
PROP. XII. THEOR.

In obtuse-angled triangles, the square of the side opposite to the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained under one of the sides about the obtuse angle upon which a perpendicular falls, and the right line without between the obtuse angle and the point whereon the perpendicular falls.

Let

Let there be an obtuse-angled triangle BAC , having the obtuse angle BAC ; and from the point B , let the perpendicular BD fall upon the continuation AD of the side CA : I say, the square of BC is greater than the squares of BA , AC by twice a rectangle contained under the right lines CA , AD .

For because the right line CD is divided any how at A , the square of CD [by 4. 2.] will be equal to the squares of CA , AD , together with twice the rectangle under CA , AD . Add the square of BD to both: then the squares of CD , DB are equal to the squares of CA , AD , BD , together with twice the rectangle under the right lines CA , AD . But [by 47. 1.] the square of CB is equal to the squares of CD , DB ; for the angle at D is a right angle; and the square of AB is equal to the squares of AD , DB : Therefore the square of CB is equal to the squares of CA , AB , and to twice the rectangle under CA , AD : Wherefore the square of CB is greater than the squares of CA , AB , by twice the rectangle contained under the right lines CA , AD .



Wherefore in obtuse-angled triangles, the square of the side opposite to the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice a rectangle contained under one of the sides containing the obtuse angle upon which a perpendicular falls, and the right line without between the obtuse angle, and the point upon which the perpendicular falls. Which was to be demonstrated.

PROP. XIII. THEOR.

In acute-angled triangles, the square of the side opposite to an acute angle is less than the squares of the sides containing that acute angle, by twice a rectangle contained under that side about the acute angle upon which a perpendicular falls, and the right line within, between the perpendicular and the acute angle^k.

Let there be an acute-angled triangle ABC , having an acute angle at B ; and from the point A draw the perpendicular

dicular AD upon BC : I say, the square of AC is less than the squares of CB , BA , by twice the rectangle contained under the right lines CB , BD .

For because the right line CB is any how divided at D , the squares of CB , BD [by 7. 2.] are equal to twice the rectangle contained under the right lines CB , BD , together with the square of DC . Add the square of AD to both: then the squares of CB , BD , DA are equal to twice the rectangle under CB , BD , together with the squares of AD , DC . But [by 47. 1.] the square of AB is equal to the squares of BD , DA , since the angle at D is a right angle; and the square of AC is equal to the squares of AD , DC : therefore the squares of CB , BA are equal to the square of AC , together with twice the rectangle contained under the right lines CB , BD . Wherefore the square of AC alone is less than the squares of CB , BA , by twice the rectangle contained under the right lines CB , BD .

Therefore in acute-angled triangles, the square of the side opposite to any acute angle, is less than the squares of the sides containing that acute angle, by twice the rectangle under that side about the acute angle upon which the perpendicular falls, and the right line within between the acute angle and the perpendicular. Which was to be demonstrated.

* Euclid has not mentioned expressly the following useful theorem: *In any obtuse-angled triangle, if a perpendicular be let fall upon the base or side opposite to the obtuse angle, the square of either of the sides containing the obtuse angle, together with twice the rectangle under the base and that segment of it (made by the falling of the perpendicular upon it) which lies under the remaining side containing the obtuse angle, will be equal to the square of this remaining side, and the square of the base, taken together.* Its demonstration is the very same as that of this 13th proposition. The theorem likewise holds true in any right-angled triangle, when the perpendicular falls from the right angle upon the base, or side opposite to it.

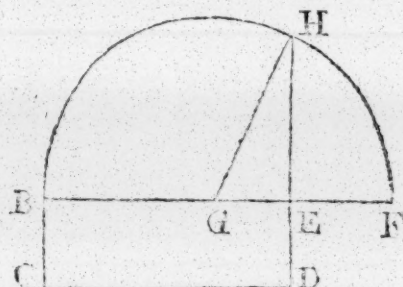
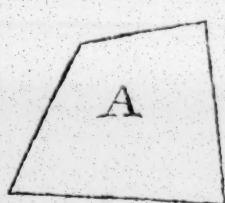
PROP.

PROP. XIV. PROBL.

To describe or make a square equal to a given right-lined figure.

Let the given right-lined figure be *A*: it is required to describe or make a square equal to that right-lined figure.

Make [by 45. 1.] the right-angled parallelogram *B D* equal to the right-lined figure *A*. If therefore *B E* be equal to *E D*, what was required, will be already



done. For the square *B D* is made equal to the right-lined figure *A*. If not, either *B E*, or *E D*, must be greater. Let *B E* be the greater, which continue out to *F*. So that *E F* [by 3. 1.] be equal to *E D*: then divide *F B* into two equal parts in *G*, and with the centre *G*, and distance equal to either *G B*, or *G F* [by post. 3.] describe a semicircle *B H F*, produce *D E* to *H*, and join *G H*.

Now because the right line *B F* is divided into two equal parts at *G*, and into unequal ones at *E*: the rectangle under *B E*, *E F*, together with the square of *E G* [by 5. 2.] will be equal to the square of *G F*. But *F G* is equal to *G H*: Wherefore the rectangle under *B E*, *E F*, together with the square of *E G*, is equal to the square of *G H*. But [by 47. 1.] the squares of *H E*, *G E* are equal to the square of *G H*: therefore the rectangle under *B E*, *E F*, together with the square of *E G*, is equal to the squares of *H E*, *G E*, take away the square of *E G* from both, then the remaining rectangle under *B E*, *E F*, is equal to the square of *E H*. But the rectangle under *B E*, *E F*, is the parallelogram *B D*, because *E F* is equal to *E D*: therefore the parallelogram *B D* is equal to the square of *H E*. But the parallelogram *B D* [by constr.] is equal to the right-lined figure *A*: therefore the right-lined figure *A* will be equal to the square of *E H*.

Wherefore the square described upon the right line *E H*, will be equal to the given right-lined figure *A*. Which was to be done.

Additions

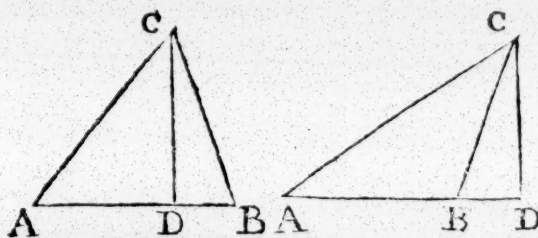
Additions to the Second Book.

PROP. I. THEOR.

If a perpendicular be drawn from the opposite angle of any triangle, upon the base, making two segments thereupon, the square of one of the sides, together with the square of that segment next to the other side, will be equal to the square of the other side, together with the square of that segment of the base next to the first mentioned side. And the difference of the squares of the sides will be equal to the difference of the squares of the said segments of the base.

Let there be any triangle ABC , and let the perpendicular CD be drawn from the angle C opposite to the base upon the base, making the two segments AD , DB of the base: I say, the square of the side AC , together with the square of the segment DB of the base, next to the side CB , will be equal to the square of the other side CB , together with the square of the segment AD of the base. And the difference of the squares of the sides AC , CB , will be equal to the difference of the squares of the segments of the base AD , DB .

For let the side AC be greater than the side BC . Then because the angle ADC is a right angle, the square of the side AC [by 47. I.] will be equal to the squares of the



segment AD of the base, and of the perpendicular CD ; also because the angle CDB is a right angle, the square of the side CB [by 47. I.] will be equal to the squares of the segment DB of the base, and of the perpendicular CD . But since, if equal things be added to equal ones, the sums will be equal. Therefore the squares of AC ,

CB ,

BD , CD will be equal to the squares of CB , AD , CD , and taking away the common square of CD from both, there will remain the squares of AC , BD , equal to the squares of CB , AD , which was one thing to be demonstrated. And because, if from equal things be taken away equal ones, the remainders are equal. Therefore the difference between the squares of AC and CB , will be equal to the difference between the squares of AD , DB . Which was the other thing to be demonstrated.

Therefore, &c. Which was to be demonstrated.

PROP. II. THEOR.

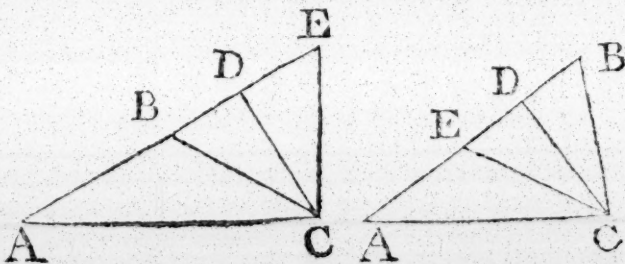
If the angle opposite to the base of a triangle be two third parts of two right angles, the square of the base of that triangle will be equal to the squares of the two sides, together with the rectangle under the two sides, and if that angle be one third part of two right angles, the square of the base will be equal to the difference of the squares of the two sides, and the rectangle under them.

Let there be a triangle ABC , having the angle ABC opposite to the base, two third parts of two right angles: I say, the squares of the sides AB , BC , together with the rectangle under the sides AB , BC , will be equal to the square of the base AC .

Continue one of the sides AB to E , and upon AE , let the perpendicular CD be drawn [by 12. 1.] from the angle ACB , make DE equal to BD , and join the right line CE .

Then because [by 13. 1.] the angles ABC , EBC are equal to two right angles; and ABC [by supposition] is equal to two third parts of two right angles; the angle EBC will be one third part of two right angles; and because the angles

BDC , EDC are right angles, for CD is perpendicular to AE ; and BD is made equal to DE ,



and

and the side CD is common. Therefore [by 4. 1.] the sides BC , CE of the right angled triangles BDC , DEC will be equal to one another; and accordingly [by 6. 1.] the angle BEC will be equal to the angle EBD . But EBD has been proved to be equal to one third part of two right angles. Therefore the angle BEC is also one third part of two right angles. And so because the three angles of every triangle are [by 32. 1.] equal to two right angles, the remaining angle BCE will be one third part of two right angles. Consequently, the triangle BEC will be equilateral; and so the sides BE , BC will be equal; and twice BD will be equal to the side BC . Again, because in the obtuse-angled triangle ABC , the perpendicular CD falls upon the continuation of the side AB ; the square of AC [by 12. 2.] is equal to the squares of AB , BC , together with twice the rectangle under AB , BD ; and because twice the rectangle under AB , BD is equal to the rectangle under AB , and twice BD ; that is, BE , and BE has been proved to be equal to the side BC . Therefore twice the rectangle under AB , BD , will be equal to the rectangle under the sides AB , BC of the triangle. Consequently, the squares of the sides AB , BC , together with twice the rectangle under AB , BD , will be equal to the squares of AB , BC , together with the rectangle under AB , BC ; and therefore the square of the base AC , will be equal to the squares of the sides AB , BC , together with the rectangle under the sides AB , BC . Which was the first thing to be demonstrated.

Secondly, Let the angle ABC of the triangle ABC opposite to the base, be one third part of two right angles: I say, the square of the base AC will be equal to the difference between the squares of the sides AB , BC , and the rectangle under them.

For the construction being the same as before, only CD being here perpendicular to the side AB , and not its continuation, we demonstrate as before, that the rectangle under the sides AB , BC , is equal to twice the rectangle under the side AB , and the segment DB . And therefore because [by the note to 13. 2.] the difference between the squares of the sides AB , BC , and twice the rectangle under the side AB , and the segment DB , is equal to the square of the base AC ; the square of the base AC , will be equal to the difference of the squares of the sides AB , BC , and the rect-

angle under them. Which was the second thing to be demonstrated.

Therefore if, &c. Which was to be demonstrated.

SCHOLIUM.

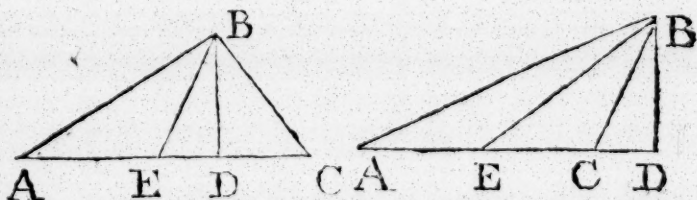
If the angle opposite to the base be one sixth part of two right angles; it is easily demonstrated by the forty-seventh proposition of the first book, that the square of the base is equal to the difference between the sum of the squares of the sides and the rectangle under them, multiplied by the square root of the number 3.

PROP. III. THEOR.

In any triangle, if the base be bisected, and a right line be drawn from the opposite angle to the point of bisection: I say, the squares of the two sides of the triangle will be equal to twice the square of the bisecting line, together with twice the square of one half the base.

Let there be a triangle ABC , whose base AC is bisected in the point E , by the right line BE drawn from the angle B opposite to the base: I say, the squares of the sides AB , BC are equal to twice the square of the bisecting line BE , together with twice the square of the half base AE , or EC .

For [by 12. 1.] from the angle B , draw the perpendicular BD upon the base AC . Then [by 47. 1.] the squares



of AB , BC are equal to the squares of AD , DC , together with twice the square of BD . But since AC is divided equally in E , and unequally in D ; the squares of AD , DC will be [by 9. 2.] equal to twice the squares of AE , ED ; therefore the squares of AB , BC will be equal to twice the squares of BD , AE , ED . But since twice the square of BE is equal to twice the squares of ED , DB . For [by 47. 1.]

H

the

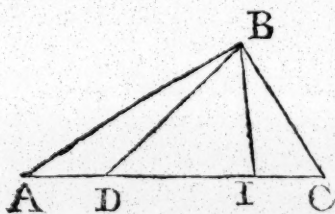
the square of EB is equal to the squares of ED , DB . Therefore the squares of AB , BC will be equal to twice the squares of AE , EB .

Again [fig. 2.] the square of AB [by 13. 2.] exceeds the squares of AE , EB by twice the rectangle under AE , ED (because the right line AC is bisected, and CD is added to it.) And the square of BC [by 12. 2.] is less than their sum by twice the rectangle under CE , ED , that is, twice the rectangle under AE , ED . Therefore if to the square of BC be added the excess whereby the square of AB exceeds the squares of AE , EB ; the squares of AB , BC , will be equal to twice the squares of AE , EB .

Wherefore &c. Which was to be demonstrated.

SCHOLIUM.

This is one of the useful and elegant propositions of the arithmeticians, to be found in Pappus's Mathematical collections, and from it may be easily obtained the following theorem. If the base AC of any triangle ABC be divided into any number of equal parts, of which AD , IC are the first and last; and if right lines BD , BI be drawn from the angle B opposite to the base joining those two remote points of division, the squares of the sides AB , BC , of the triangle together, are equal to the squares of the right lines BD , BI , together with the square of one of the parts AD , or IC , of the base, as many times taken wanting two, as there are unites in twice the number of parts the base is divided into.



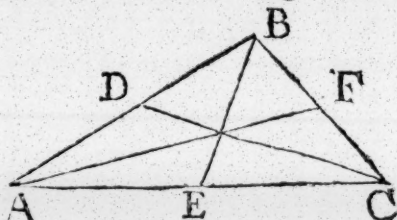
PROP. IV. THEOR.

If the three sides of any triangle be bisected, and right lines be drawn from the opposite angles to the points of bisection, four times the sum of the squares of these three bisecting right lines will be equal to thrice the sum of the squares of the sides of the triangle.

Let there be any triangle ABC , whose three sides are bisected in the points D , E , F , to which are drawn the right

right lines AF , BE , CD from the opposite angles: I say four times the sum of the squares of AF , BE , CD will be equal to thrice the sum of the squares of the sides AB , AC , CB of the triangle.

For since [by 3. of this] twice the squares of AF , CF is equal to the squares of the sides AB , AC of the triangle, and twice the squares of EB , AE is equal to the squares of the sides AB , BC of the triangle, and twice the square of CD , AD is equal to the squares of the sides AC , CB of the triangle. There-



fore twice the sums of the former squares, that is, twice the sum of the squares of AF , CF , EB , AE , CD , AD will be equal to twice the sum of the squares of AB , CB , AC . And so the sum of the squares of AF , CF , EB , AE , CD , AD will be equal to the sum of the squares of AB , CB , AC . And since [by 4. 2.] the square of any line is equal to four times the square of one half of it, four times the sum of the squares of CF , AE , AD is equal to the sum of the squares of CB , AC , AB ; therefore the sum of the squares of CF , AE , AD will be equal to one fourth part of the sum of the squares of CB , AC , AB . And so the sum of the squares of AF , EB , CD , together with one fourth part of the sum of the squares of CB , AC , AB will be equal to the sum of the squares of CB , AC , AB . And taking away one fourth part of the sum of the squares of CB , AC , AB from both, the sum of the squares of AF , EB , CD will be equal to three fourth parts of the sum of the squares of CB , AC , AB . And taking four times each sum, four times the sum of the squares of AF , EB , CD will be equal to three times the sum of the squares of CB , AC , AB .

Therefore, &c. Which was to be demonstrated.

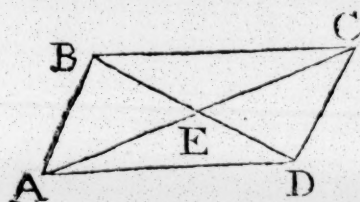
PROP. V. THEOR.

In every parallelogram the squares of the sides taken together, are equal to the squares of the diagonals.

Let there be any parallelogram $ABCD$ whose diagonals are AC , DB : I say the squares of the sides AB , BC , CD , AD together, will be equal to the two squares of the diagonals AC , BD .

For let E be the point of intersection of the diagonals.

Then because AB, DC are parallel, the alternate angles ABE, EDC , are [by 29. I.] equal; and because AD, BC are parallel, the alternate angles BAE, ECD are also equal. And since the sides AB, DC also are equal; there are two



triangles ABE, DEC having two angles of the one respectively equal to two angles of the other, and the sides between the equal angles also equal. Wherefore [by 26. I.] the other sides will

also be equal, and the remaining angles. Wherefore BE will be equal to ED , and AE equal to EC . Therefore the diagonals of the parallelogram $ABCD$ do mutually bisect one another. And so because [by 4. of this] the squares of AB, BC are equal to twice the squares of AE, EB , and the squares of AD, DC equal to twice the squares of AE, ED , therefore the sum of the squares of AB, BC, AD, DC will be equal to twice the sum of the squares of AE, BE, AE, ED . That is, equal to four times the square of AE , together with twice the squares of BE, ED . That is [since BE has been proved to be equal to ED] to four times the square of AE , and four times the square of BE . But [by 4. 2.] the square of AC is equal to four times the square of AE , and the square of BD equal to four times the square of BE . Therefore the squares of the sides of AB, BC, CD, AD will be equal to the squares of the diagonals AC, BD .

Therefore, &c. Which was to be demonstrated.

SCHOLIUM.

The following theorem in a six-sided parallelogram is likewise true, viz. that the sum of the squares of the sides is equal to the difference between twice the sum of the squares of the three diagonals joining the opposite angles, or intersecting one another in one point, and the sum of the squares of the six remaining diagonals. There are also other such like properties of eight-sided parallelograms, &c. but these I leave for the discovery of others.

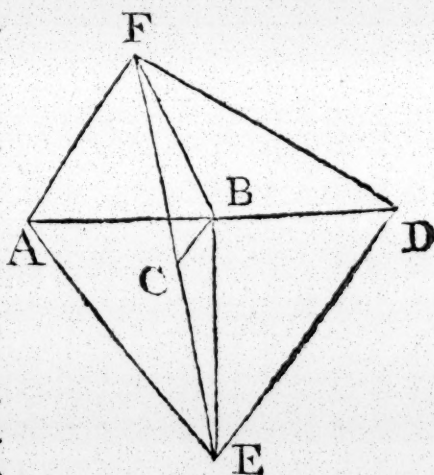
PROP.

PROP. VI. THEOR.

In any quadrilateral figure, the sum of the squares of the four sides, will be equal to the sum of the squares of the diagonals, together with four times the square of the right line joining the middle points of the diagonals.

Let there be a quadrilateral figure $A F D E$, and let its diagonals AD , FE be bisected in the points B , C : I say the sum of the squares of the four sides AF , FD , AE , ED will be equal to the sum of the squares of the diagonals AD , FE together with four times the square of the right line BC joining the points of bisection B , C of the diagonals.

For join the right lines FB , BE . Then because the base AD of the triangle $A F D$ is bisected in B ; the squares of AF , FD [by 4. of this] are equal to twice the squares of AB , BF . And likewise the squares of the sides AE , ED of the triangle $A E D$ equal to twice the squares of AB and BE . Therefore the sum of the squares of AF , FD , AE , ED will be equal to four times the square of AB , together with twice the squares of FB , BE . Or since [by 4. 2.] four times the square of AB is equal to the square of the diagonal AD , the sum of the



squares of the four sides of the figure will be equal to the square of the diagonal AD , together with twice the squares of FB , BE . But because the side FE of the triangle $F B E$ is bisected in C , the squares of FB , BE [by 4. of this] are equal to twice the squares of FC , and CB ; and so twice the squares of FB , BE are equal to four times the square of FC , and four times the square of CB . Therefore the sum of the squares of the sides will be equal to the square of the diagonal AD , together with four times the square of FC , and four times the square of CB ; that is, since the square of the diagonal FE is [by 4. 2.] four times the square of

7 c) the sum of the squares of the sides AF , FD , AE , ED will be equal to the sum of the squares of the diagonals AD , EF , together with four times the square of the right line BC joining the middle points B , C of the diagonals.

Therefore &c. Which was to be demonstrated.

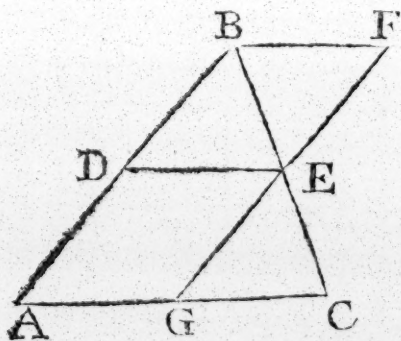
LEMMA.

If the two sides of a triangle be bisected, a right line joining the points of bisection will be parallel to the base of the triangle, and equal to one half of it.

Let there be a triangle ABC , whose sides AB , CB , are bisected in the points D , E : I say the right line DE joining the points D , E will be parallel to the base AC of the triangle, and equal to one half of it.

For draw [by 31. 1.] BF parallel to AC , and thro' B draw GF parallel to the side AB meeting the base AC in the point G , and the right line BF in the point F .

Now because [by 15. 1.] the angle BEF is equal to the angle GEC , and the angle EBF [by 29. 1.] is equal to the angle GCE , and the side BE [by supposition] is equal to the side EC . Therefore [by 26. 1.] the right line BF will be equal to GC . But [by 34. 1.] AG is equal to BF , and ABE equal to GF . Wherefore BD will be equal to FE , that is, equal to EG .



Again, because BE is equal to EC , and BD equal to EG , and [by 29. 1.] the angle DBE is equal to the angle GEC . Therefore [by 4. 1.] DE will be equal to GC ; that is, equal to BF , or equal to AG , and the angle BED will be equal to the angle ACB . Therefore [by 28. 1.] DE will be parallel to AC . And it has been proved to be equal to AG or GC . Wherefore DE will also be equal to one half the base AC . Which was to be demonstrated.

PROP

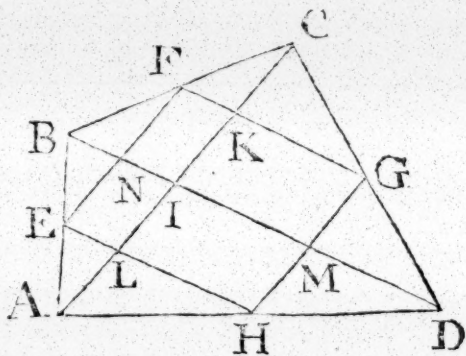
PROP. VI. THEOR.

If the four sides of any trapezium be bisected, and the points of bisection be joined by four right lines, the figure contained under these right lines will be a parallelogram, and the parallelogram will be one half of the trapezium.

Let $ABCD$ be a trapezium, whose sides AB, BC, CD, AD are bisected in the points E, F, G, H , and the right lines EF, FG, GH, EH . Join the points of bisection: I say the figure $EFGH$ contained under them will be a parallelogram, and this parallelogram will be one half of the trapezium $ABCD$.

For draw the diagonals AC, BD .

Then since the sides AB, BC of the triangle ABC , are bisected in the points E, F , the right line EF joining the points of bisection will [by the precedent lemma] be parallel to the diagonal AC . In like manner the right line HG will be parallel to AC . But right lines that are parallel to the same right line, are [by 30. 1.] parallel to one another. Therefore the right line EF will be parallel to the right line HG . By the same reason the right line FG will be parallel to the right line HE : Therefore the figure $EFGH$ is a parallelogram.



Again, Let the point I be the intersection of the diagonals. Let EF, HG cut the diagonal BD in the points N, M , and the lines FG, EH the diagonal AC in the points K, L . Then because FN, GM are parallel, and CK is equal to KI ; therefore [by 41. 1.] the parallelogram NK will be double to the triangle FCK . In like manner, because FG is parallel to BD , and BN is equal to NI , the parallelogram NK will be double to the triangle BFN . And so the parallelogram NK will be equal to both the triangles BFN, FCK . So also will the parallelogram IG be

H 4

equal

equal to the triangles CKG , GMD . Wherefore the parallelogram NG will be equal to one half the triangle BCD . By the same reason, the parallelogram EM will be equal to one half the triangle ABD . Wherefore the two parallelograms NG , EM together; that is, the parallelogram $EFGH$ will be one half of the sum of the two triangles BCD , ABD ; that is, one half the trapezium $ABCD$.

Therefore, &c. Which was to be demonstrated.

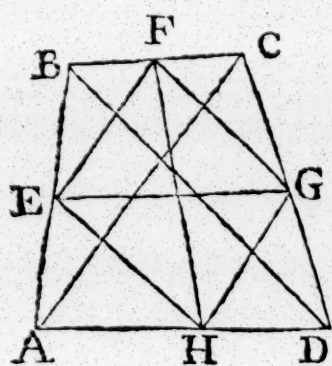
PROP. VIII. THEOR.

If the middle points of the opposite sides of any quadrilateral figure be joined by two right lines, the sum of the squares of these two right lines will be one half the sum of the squares of the diagonals of the quadrilateral figure.

Let $ABCD$ be a quadrilateral figure whose sides are bisected in the points E , F , G , H , which are joined by the right lines EG , FH : I say the sum of the squares of the right lines EG , FH , will be equal to one half the sum of the squares of the diagonals AC , BD of the quadrilateral figure.

For join the right lines EF , FG , GH , EH .

Then because the sides of the quadrilateral figure are bisected in the points E , F , G , H , [by 7. of this] the figure



$EFGH$ will be a parallelogram: And the equal opposite sides EF , GH will be each of them one half of the diagonal AC of the quadrilateral figure, and the equal opposite sides FG , EH each of them equal to one half the diagonal BD , of the quadrilateral figure. Therefore [by 4.2.] the sum of the squares of the four sides EF , FG , GH , EH of the parallelogram $EFGH$ will be equal to one half the sum of the squares of the diagonals AC , BD of the quadrilateral figure $ABCD$. But [by 5. of this] the sum of the squares of the diagonals EG , FH of the parallelogram; that is, of the right lines joining the middle points of the sides of the quadrilateral figure $ABCD$, is equal to the sum of the squares

of

of the sides of the parallelogram. Therefore the sum of the squares of the right lines EG , FH , will be equal to one half the sum of the squares of the diagonals AC , BD , of the trapezium $ABCD$.

Therefore &c. Which was to be demonstrated.

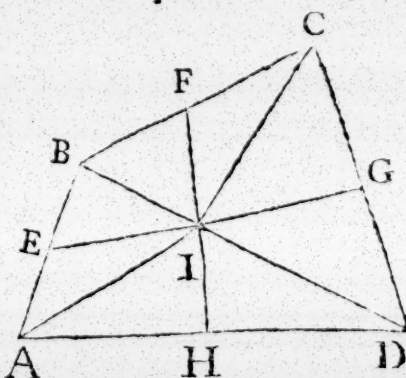
PROP. IX. THEOR.

If the middle points of the opposite sides of any trapezium be join'd by two right lines, the sum of the squares of two opposite sides, together with twice the square of the right line joining the other two opposite sides, will be equal to the sum of the squares of the other two opposite sides, together with twice the square of the right line joining the middle points of the two opposite sides first assumed.

Let there be a trapezium $ABCD$, whose sides are bisected in the points E , F , G , H . And let the right lines EG , FH , be drawn. I say, the sum of the squares of the opposite sides AB , CD of the trapezium, together with twice the square of the right line FH , will be equal to the sum of the squares of the other two opposite sides AD , BC , together with twice the square of the right line EG .

For let the point I be the intersection of the right lines EG , FH joining the points of bisection of the opposite sides of the trapezium; and join AI , BI , CI , DI .

Then because the side AD of the triangle AID is bisected in the point H , the squares of AI , ID will [by 5. of this] be equal to twice the squares of AH , HI . In like manner since AB is bisected in E ; the squares of AI , BI will be equal to twice the squares of AE , EI . So likewise will the squares of BI , IC be equal to twice the squares of BF , FI , and the squares of CI , ID equal to twice the squares of CG , IG . Therefore the squares of AI , BI , CI , DI will be equal to twice the squares of AH , HI , BF , FI , and also the squares of AI , BI , CI , DI equal to twice the



the squares of AE, EI, CG, IG . Wherefore twice the squares of AH, HI, BF, FI will be equal to twice the squares of AE, EI, CG, IG . And [by doubling each sum] four times the squares of AH, HI, BF, FI will be equal to four times the squares of AE, EI, CG, IG . But [by 4. 2.] four times the square of AH is equal to the square of AD , four times the square of BF is equal to the square of BC , and four times the square of FI, IH is equal to twice the square of FH . In like manner four times the square of AE is equal to the square of AB , four times the square of CG equal the square of CD , and four times the square of EI, IG equal to twice the square of EG . Wherefore [by 1. 2.] the squares of AD, BC , together with twice the square of EG , will be equal to the squares of AB, CD , together with twice the square of FH .

Wherefore &c. which was to be demonstrated.

SCHOLIUM.

The following theorem is also easily obtained and demonstrated, viz. That in any trapezium, if the sides be bisected, and two right lines join the opposite points of bisection, and if four right lines be drawn from their intersection to the angles of the trapezium; four times the sum of the squares of these right lines will be equal to the sum of the squares of the four sides of the trapezium, together with twice the sum of the squares of the two right lines joining the middle points of the sides of the trapezium.

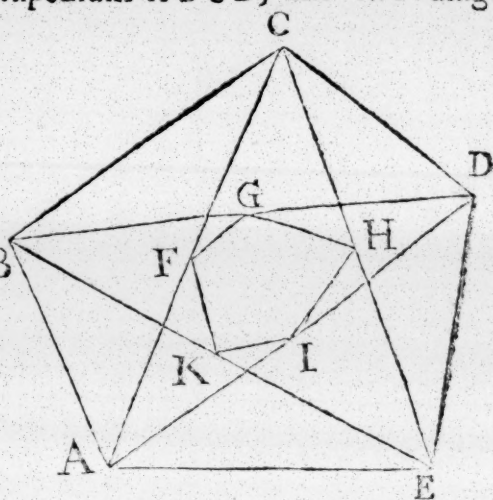
PROP. X. THEOR.

In any five-sided right-lined figure, thrice the sum of the squares of the sides will be equal to the sum of the squares of the diagonals, together with four times the sum of the squares of the five right lines orderly joining the middle points of the diagonals.

Let there be a five-sided right lined figure $ABCDE$; and let its five diagonals AC, BD, CE, DA, EB be bisected in the points F, G, H, I, K . And let each two of these middle points next to one another be joined by the five right lines FG, GH, HI, IK, KF : I say, thrice the sum of the squares of the sides AB, BC, CD, DE, EA of the figure

figure will be equal to the sum of the squares of the diagonals AC , BD , CE , DA , EB , together with four times the sum of the squares of the five right lines FG , GH , HI , IK , KF .

For because in the trapezium $ABCD$, the two diagonals AC , BD are bisected in the points F , G , and the right line FG joins them. Therefore [by 6. of this] the sum of the squares of AB , BC , CD , AD is equal to the sum of the squares of AC , BD , together with four times the square of FG . In like manner, in the trapezium $BCDE$, the



sum of the squares of BC , CD , DE , BE is equal to the sum of the squares of BD , CE , together with four times the square of GH . And in the trapezium $CDEA$, the sum of the squares of CD , DE , EA , AC is equal to the sum of the squares of CE , AD , together with four times the square of HI . Also in the trapezium $ABDE$, the sum of the squares of DE , EA , BA , BD is equal to the sum of the squares of AD , BE together with four times the square of IK . And in the trapezium $EABC$, the sum of the squares of AE , AB , BC , CE is equal to the squares of AC , BE , together with four times the square of KF . And adding all the former squares together, their sum will be equal to three times the square of each side AB , BC , CD , DE , EA together with the sum of the squares of the five diagonals AD , BE , AC , BD , CE . Also adding all the latter squares together, their sum will be equal to twice the squares of each of the five diagonals AC , BD , CE , AD , BE , together with four times the squares of the right lines FG , GH , HI , IK , KF . But since the sum of all the former squares is equal to the sum of all the latter: And if from both these sums be taken away the sum of the squares of all the diagonals, there will remain thrice the squares of each side AB , BC , CD , DE , EA equal to the sum of the squares of the diagonals AC , BD , CE , AD , BE , together with four times the squares

squares of each right line FG , GH , HI , IK , KF ; that is, thrice the sum of the squares of the sides AB , BC , CD , DE , EA of the five-sided right lined figure will be equal to the sum of the squares of the diagonals, together with four times the sum of the squares of the right lines orderly joining the middle points of the diagonals.

Therefore &c. Which was to be demonstrated.

SCHOLIUM.

There are other the like properties of right-lined figures of six or more sides. In all which some number of times the sum of the squares of the sides will be equal to some number of times the sum of the squares of the diagonals together with some number of times the squares of the proper right lines joining the middle points of the diagonals of the figures.

E U C L I D's E L E M E N T S. B O O K III.

D E F I N I T I O N S.

1. **E**QUAL circles are those whose diameters, or the right lines that be drawn from their centres, are equal ^a.
2. A right line is said to touch a circle when meeting it and being produced does not cut it.
3. Circles are said to touch one another, which meeting one another, do not cut one another.
4. Right lines in a circle are said to be equally distant from the centre, when perpendiculars drawn to them from the centre are equal.
5. And that line is said to be farther from the centre, upon which the greater perpendicular falls.
6. A segment of a circle is a figure contained under a right line and a part of the circumference of a circle ^b.
7. An angle of a segment is that which is contained under a right line and the circumference of a circle ^c.
8. An angle in a segment, is the angle contained under two right lines drawn from any assumed point in the cir-

^a This is rather an axiom than a definition ; for what else is it but taking for granted this proposition, *viz.* those circles are equal, whose diameters or semidiameters are equal.

^b From hence it follows that the segment of a circle is three-fold, *viz.* a semicircle when the right line passes through the centre, a greater segment, and a lesser.

^c This definition seems to be of no great use, at least in these elements.

cumference

cumference of the circle, to the ends of the line, being the base of that segment.

9. But when the right lines containing such an angle do cut off part of the circumference of the circle, the angle is said to stand upon that part cut off.

10. A sector of a circle is the figure comprehended between two right lines drawn from the centre of the circle and that part of the circumference of the circle which is between them.

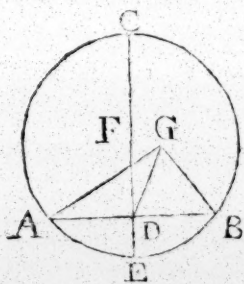
11. Similar segments of circles are those which receive equal angles, or whereof the angles in them are equal.

PROPOSITION I. PROBLEM.

To find the centre of a given circle^d.

Let ABC be a given circle: it is required to find the centre of the circle ABC .

Let any right line AB be drawn in it; and [by 10. 1.] divide the same in two equal parts at D ; and draw DC [by 11. 1.] at right angles to AB , which produce to E . Then divide CE into two equal parts at F : I say the point F is the centre of the circle ABC .



For if it be not, let, if possible, G be the centre, and draw GA , GD , GB . Then because AD is equal to DB , and DG is common, the two sides AD , DG will be equal to the two sides GD , DB each to each. And [by def. 15. 1.] the base GA is equal to the base GB , for

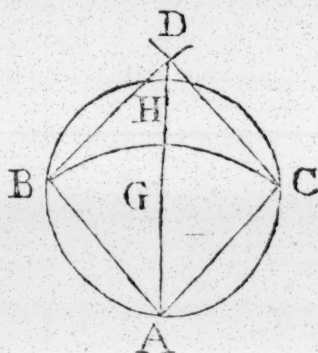
they are both drawn from the centre G . Therefore the angle ADG [by 8. 1.] is equal to the angle GDB . But since a right line standing upon a right line makes the adjoining angles equal to one another; each of these angles [by 10. def. 1.] are right angles: Therefore the angle GDB is a right angle. But FDB is a right angle too: Wherefore the angle FDB is equal to the angle GDB , the greater equal to the lesser, which is impossible. Therefore G is not the centre of the circle ABC . After the same manner we demonstrate that no other point besides F can be the centre.

Therefore the point F is the centre of the circle ABC . Which was to be done.

Corol.

Corol. From hence it is evident, If any right line in a circle cuts another into two equal parts, at right angles, the centre of the circle will be in the cutting line.

^d The praxis of this problem may be somewhat shorter, thus: Open the compasses to any convenient distance that will cut the given circle, and setting one foot in the circumference at the point A describe an arch BC with the other foot, cutting the circumference of the given circle in the points B, C, and about the centres B, C, with the distance AB or AC describe two arches intersecting one another in the point D. Draw the right line AD cutting the circumference of the circle in H. Bisect AH in G, and then will G be the centre of the given circle. The demonstration cannot be here given, because it depends upon the xxivth proposition of this book.



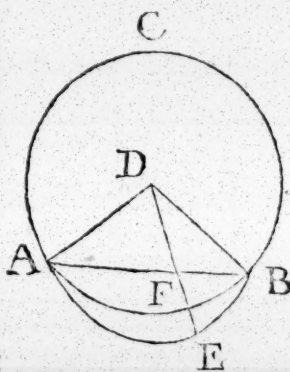
PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the right line which joins them, will fall within the circle^e.

Let ABC be a circle, in whose circumference let any two points A, B be taken: I say the right line AB joining those two points, falls within the circle.

For if not, let it fall without, if possible as AEB; and [by 1. 3.] find the centre D, of the circle ABC, join AD, DB, and continue out DF to E.

Then because DA is equal to DB, the angle DAE [by 5. 1.] will be equal to the angle DBE; and because one side AEB of the triangle DAE is produced, the angle DEB [by 16. 1.] will be greater than the angle DAE. But the angle DAE is equal to the angle DBE: Therefore the angle DEB will be greater than the angle DBE. But [by 18. 1.] the greater side is opposite to the greater angle. Therefore DB is greater than DE. But DF is equal to the right line DB: Wherefore DF is greater than



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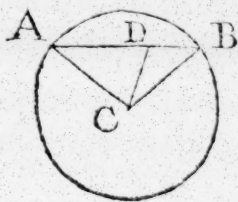
A B C.

Corol.

than DE , the lesser than the greater, which is impossible. Therefore the right line drawn from the point A to B does not fall without the circle, nor does it fall in the circumference. Consequently it must necessarily fall within the circle.

If therefore any two points be taken in the circumference of a circle, the right line which joins them will fall within the circle. Which was to be demonstrated.

* This theorem may be demonstrated affirmatively thus: Let the right line AB touch two points A, B in the circumference of the circle ABC , whose centre is C : I say the right line AB falls within the circle, so that all its intermediate points will be within the circle. For take any one of its intermedi-



ate points D , and from the centre draw the right lines CA, CD, CB . Then because the two sides CA, CB of the triangle CAB are equal; the angles CAB, CBA [by 5. 1.] will be equal to one another. But [by 16. 1.] the angle CDA is greater than the angle CBA , the external than the internal. There-

fore the same angle CDA will be greater than the angle CAD , and so the side CA will be [19. 1.] greater than the side CD . Wherefore since CA extends from the centre to the circumference; the right line CD will not extend to the circumference. Consequently the point D will fall within the circle. The same may be demonstrated of any other intermediate point of the right line AB . Wherefore the whole right line AB falls within the circle.

Some have thought this proposition needs no demonstration, it being so very self evident. But it must either be a demonstrable proposition, or an axiom or indemonstrable one. Euclid did not care to make it an axiom, because of increasing their number too much; and therefore did rightly in demonstrating it.

PROP. III. THEOR.

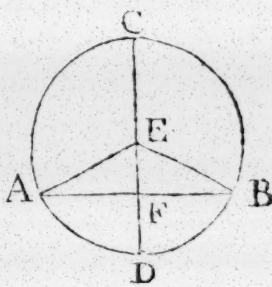
If a right line drawn thro' the centre of a circle divides a right line not drawn thro' the centre into two equal parts, it will cut the same at right angles: and if it cuts the same at right angles, it will cut it into two equal parts.

Let there be a circle ABC , and let the right line CD drawn in it thro' the centre, divide the right line AB not drawn

drawn thro' the centre into two equal parts: I say it will cut the same at right angles.

For [by 1. 3.] find the centre of the circle ABC , which let be E , and join EA , EB .

Then because AF is equal to FB , and FE is common, two sides AF , FE will be equal to two sides BF , FE , and the base EA is equal to the base EB ; therefore the angle AFE [by 8. 1.] will be equal to the angle BFE . But since a right line standing upon a right line makes the adjoining angles equal to one another, each of the equal angles is a right angle: therefore each of the angles AFE , BFE is a right angle, and so the right line CD drawn thro' the centre cutting the right line AB , not drawn thro' the centre, into two equal parts, cuts it at right angles.



Now if the right line CD cuts the right line AB at right angles: I say it will divide the same into two equal parts AF , BF .

For the same construction remaining, because the right line EA drawn from the centre, is equal to EB , the angle EAF will be [by 5. 1.] equal to the angle EBF . But the right angle AFE is equal to the right angle BFE : therefore the two triangles EAF , EBF have two angles of the one equal to two angles of the other, and the side EF of the one equal to the side EF of the other, viz. the common side opposite to one of the equal angles: therefore the remaining sides of the one triangle will be [by 26. 1.] equal to the remaining sides of the other: Wherefore AF is equal to BF .

If therefore a right line drawn through the centre of a circle divides a right line not drawn through the centre into two equal parts; it will cut the same at right angles: and if it cuts the same at right angles, it will cut it into two equal parts. Which was to be demonstrated.

PROP. IV. THEOR.

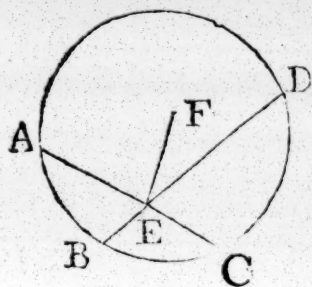
In a circle if two right lines not drawn through the centre, do intersect each other; they will not cut one another into two equal parts.

I

Let

Let there be a circle $ABCD$, and in it let two right lines AC , BD not drawn through the centre intersect one another in the point E . I say these will not cut one another into two equal parts.

For, if possible, let them bisect each other so that AE be equal to EC , and BE equal to ED . Find [by I. 3.] the centre F of the circle $ABCD$, and join FE . Then because the right line FE drawn thro' the centre cuts the right line AC not drawn through the centre into two equal parts; it will [by 3. 3.] cut the same at right angles: wherefore FEA will be a right angle. Also because the right line FE cuts the right line BD not drawn through the centre into



two equal parts, it will also cut it at right angles: therefore FEB is a right angle. But it has been also proved that FEA is a right angle. Therefore the angle FEA will be equal to the angle FEB , the lesser equal to the greater, which is impossible.

Therefore AC , BD do not cut one another into equal parts.

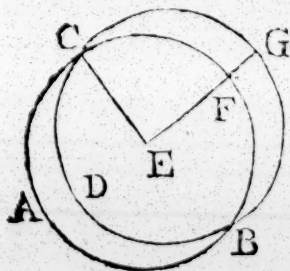
Wherefore if in a circle two right lines not drawn through the centre do intersect one another, they will not cut one another into two equal parts. Which was to be demonstrated.

PROP. V. THEOR.

If two circles intersect one another, they will not both have the same centre.

For let two circles ABC , CDG intersect one another in the points B , C : I say they have not both the same centre.

For, if possible, let E be the centre of them both. Join EC , and draw any right line EF .



Then because E is the centre of the circle ABC , the right line EC will be [by 15. def. 1.] equal to EF ; and because E is the centre of the circle CDG , the right line EC will be equal to EG . But EC has been proved to be equal to EF , wherefore EF will be

be equal to EG , the lesser equal to the greater, which is impossible. Therefore the point E is not the centre of both the circles ABC , CDG . Which was to be demonstrated.

Therefore if two circles cut one another, they will not both have the same centre.

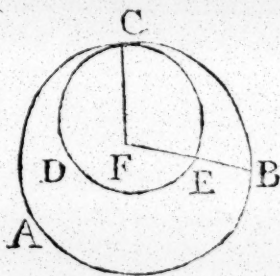
PROP. VI. THEOR.

If two circles touch one another inwardly, they will not both have the same centre^f.

For let two circles ABC , CDE touch one another inwardly in the point C . I say they have not both the same centre.

For if they have, let the same be F . Join FC , and draw FB any how.

Then because F is the centre of the circle ABC , the line FC is equal to FB . Also because F is the centre of the circle CDE , the line FC will be equal to FE : but FC has been proved to be equal to FB : therefore FE is equal to FB , the lesser to the greater, which is impossible. Wherefore F is not the centre of both the circles ABC , CDE .



Therefore if two circles touch one another inwardly, they will not both have the same centre. Which was to be demonstrated.

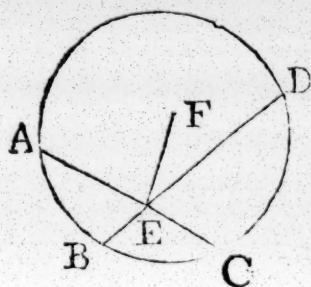
^f Euclid only proposed this theorem of circles touching one another inwardly; for when they touch outwardly, it is most manifest, they will not both have the same centre.

PROP. VII. THEOR.

If any point except the centre be taken in the diameter of a circle, and any right lines drawn from it do fall upon the circumference of the circle: the greatest of them will be that wherein is the centre; and the least, the remainder of the same line; and of all the other lines, that which is nearest the line passing

Let there be a circle $ABCD$, and in it let two right lines AC , BD not drawn through the centre intersect one another in the point E . I say these will not cut one another into two equal parts.

For, if possible, let them bisect each other so that AE be equal to EC , and BE equal to ED . Find [by I. 3.] the centre F of the circle $ABCD$, and join FE . Then because the right line FE drawn thro' the centre cuts the right line AC not drawn through the centre into two equal parts; it will [by 3. 3.] cut the same at right angles: wherefore FEA will be a right angle. Also because the right line FE cuts the right line BD not drawn through the centre into



two equal parts, it will also cut it at right angles: therefore FEB is a right angle. But it has been also proved that FEA is a right angle. Therefore the angle FEA will be equal to the angle FEB , the lesser equal to the greater, which is impossible.

Therefore AC , BD do not cut one another into equal parts.

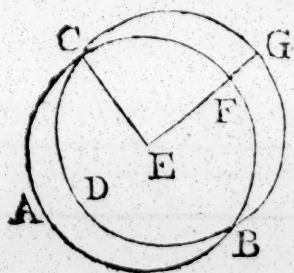
Wherefore if in a circle two right lines not drawn thro' the centre do intersect one another, they will not cut one another into two equal parts. Which was to be demonstrated.

PROP. V. THEOR.

If two circles intersect one another, they will not both have the same centre.

For let two circles ABC , CDG intersect one another in the points B , C : I say they have not both the same centre.

For, if possible, let E be the centre of them both. Join EC , and draw any right line EF .



Then because E is the centre of the circle ABC , the right line EC will be [by 15. def. 1.] equal to EF ; and because E is the centre of the circle CDG , the right line EC will be equal to EG . But EC has been proved to be equal to EF , wherefore EF will be

be equal to EG , the lesser equal to the greater, which is impossible. Therefore the point E is not the centre of both the circles ABC , CDG . Which was to be demonstrated.

Therefore if two circles cut one another, they will not both have the same centre.

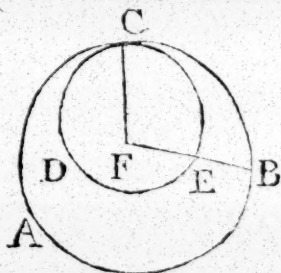
PROP. VI. THEOR.

If two circles touch one another inwardly, they will not both have the same centre^f.

For let two circles ABC , CDE touch one another inwardly in the point C . I say they have not both the same centre.

For if they have, let the same be F . Join FC , and draw FB any how.

Then because F is the centre of the circle ABC , the line FC is equal to FB . Also because F is the centre of the circle CDE , the line FC will be equal to FE : but FC has been proved to be equal to FB : therefore FE is equal to FB , the lesser to the greater, which is impossible. Wherefore F is not the centre of both the circles ABC , CDE .



Therefore if two circles touch one another inwardly, they will not both have the same centre. Which was to be demonstrated.

^f Euclid only proposed this theorem of circles touching one another inwardly; for when they touch outwardly, it is most manifest, they will not both have the same centre.

PROP. VII. THEOR.

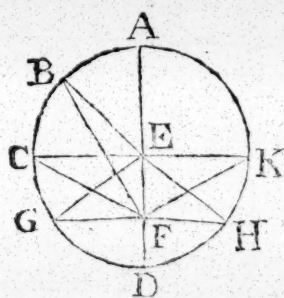
If any point except the centre be taken in the diameter of a circle, and any right lines drawn from it do fall upon the circumference of the circle: the greatest of them will be that wherein is the centre; and the least, the remainder of the same line; and of all the other lines, that which is nearest the line passing

passing through the centre will always be greater than that more remote; and only two equal lines can fall from the aforesaid point, being one on one side of the least line, and one on the other.

Let $ABCD$ be a circle, AD a diameter of it, and let any point F be taken in that diameter, not being the centre of the circle; let E be the centre of the circle, and let any right lines FB , FC , FG drawn from the point F fall upon the circumference of the circle $ABCD$: I say FA is the greatest of these right lines, and FD is the least; and of the rest, FB is greater than FC , and FC than FG .

For join BE , CE , GE .

Then because two sides of every triangle are [by 20. 1.] greater than the remaining side, the sides EB , EF will be greater than BF . But AE is equal to BE : therefore BE , EF are equal to AF : wherefore AF is greater than BF .



Again, because BE is equal to CE , and FE is common, the two sides BE , EF are equal to the two sides CE , EF . But the angle BEF is greater than the angle CEF : therefore [by 24. 1.] the base BF is greater than the base CF . By the same reason, CF is greater than FG .

Again, because GF , FE together, are greater than EG , but EG is equal to ED , the sides GF , FE will be greater than ED . Take away FE , which is common: then the remainder GF is greater than the remainder FD . Therefore FA is the greatest of the right lines, and FD the least: but FB is greater than FC , and FC than FG . I say moreover there can but two equal right lines fall from the point F upon the circumference of the circle $ABCD$, one on one side FD the least of the lines, and the other on the other side of this line. For [by 23. 1.] make the angle FEH at the given point E in the right line ED , equal to the angle GEF , and draw FH ; then because GE is equal to EH , and EF is common, the two sides GE , EF are equal to the two sides HE , EF , and the angle GEF [by constr.] is equal to the angle HEF : therefore the base FG [by 4. 1.] will be equal to the base FH . I say, no other right line can fall from F upon the circumference of the circle equal

equal to FG ; for if possible, let FK fall upon it, equal to FG ; then because FK is equal to FG , and FH is equal to FG : FK will be equal to FH too, viz. a line nearer to the line drawn thro' the centre, equal to a line which is more remote. Which is impossible.

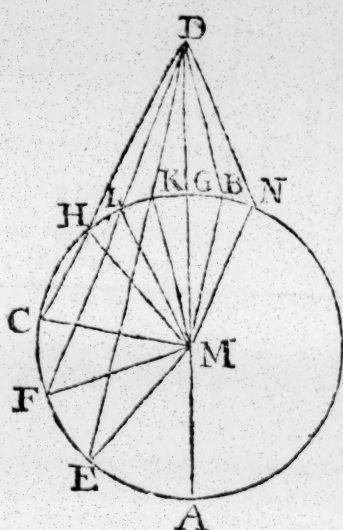
Or thus: Join EK , and because GE is equal to EK , and FE is common, and the base GF is equal to the base FK ; the angle GEF [by 8. 1.] will be equal to the angle KEF . But the angle GEF is equal to the angle HEF : therefore the angle HEF , will be equal to the angle KEF , the lesser equal to the greater. Which cannot be: wherefore no other line can fall from the point F upon the circumference of the circle equal to GF . There is therefore but one only equal to it.

If therefore any point except the centre, be taken in the diameter of a circle, &c. Which was to be demonstrated.

PROP. VIII. THEOR.

If any point be taken without a circle, and from that point be drawn any right lines to the circle, one of which passes thro' the centre, and the others any how at pleasure: that passing, thro' the centre will be the greatest of all those that fall upon the concave part of the circumference of the circle, and of the rest that which is nearest it will always be greater than that which is more remote: but amongst these lines which fall upon the convex part of the circumference, the least is that which lies between the assumed point and the diameter: of the others that which is nearer to the least will always be less than that which is more remote; and only two equal right lines can fall upon the circumference of the circle, the one on one side the least line, and the other on the other side of it.

Let ACB be a circle, and any point D be taken without it, from which any right lines DA , DE , DF , are drawn to the circumference of the circle, whereof DA is that passing thro' the centre: I say, the greatest of these lines that fall upon the concave part of the circumference



of the circle, is DA , which passes thro' the centre, and amongst the rest that will always be the greater which is the nearer to that passing thro' the centre, viz. DE greater than DF , and DF than DC . But of those lines which fall upon the convex part $HLKG$ of the circumference, DG , which lies between the diameter and the point D , is the least, and that line which is nearer to this, is always less than that which is more remote, viz. DK less than DL , and DL less

than DH .

For [by I. 3.] find the centre M of the circle ABC , and join ME , MF , MC , MK , ML , MH .

Now because AM is equal to EM , let MD , which is common be added; and then AD is equal to EM , MD . But [by 20. I.] EM , MD are greater than ED : therefore AD is greater than ED . Again, because ME is equal to MF , and MD is common, EM , MD , will be equal to MF , MD , and the angle EMD is greater than the angle FMD ; therefore the base ED [by 24. I.] will be greater than the base FD . After the same manner we prove, that FD is greater than CD ; therefore DA is the greatest of all those lines falling from D upon the circle; DE is greater than DF , and DF greater than DC .

Again, because MK , KD [by 20. I.] are greater than MD , and MG is equal to MK , the remainder KD will be greater than the remainder GD : wherefore GD is less than KD , and so it is the least of them all, and because two right lines MK , KD stand upon one side MD of the triangle MLD within it, MK , KD [by 21. I.] will be less than ML , LD ; but MK is equal to ML . Therefore the remainder DK is less than the remainder DL . In like manner we demonstrate that DL is less than DH , therefore DG is the least of these right lines, DK is less than DL , and DL less than DH .

I say also, that only two equal right lines fall from D upon the circumference of the circle, the one on one side, and the other on the other side of the least line; for [by 23. I.] at the given point M in the right line MD make the angle

angle DMB with it equal to the angle KMD , and join DB . then because MK is equal to MB , and MD is common, the two right lines KM , MD , are equal to the two right lines BM , MD , each to each, and the angle KMD is equal to the angle $BM D$: therefore [by 4. 1.] the base KD is equal to the base DB . I say also that no other right line can fall from D , upon the circumference of the circle equal to DK . For if there can, let it be DN . And because DK is equal to DN , and DB equal to DK ; the right line DB will also be equal to DN , that is, the line which is the nearer equal to that which is more remote. Which has been demonstrated to be impossible.

Or thus: Join NM , and because KM is equal to MN , and MD is common, and the base DK is equal to the base DN ; the angle KMD , [by 8. 1.] will be equal to the angle DMN , but the angle KMD is equal to the angle $BM D$. Therefore the angle $BM D$ will be equal to the angle NMD , the lesser equal to the greater: which is impossible. Wherefore there cannot fall from the point D upon the circumference of the circle ABC , more than two equal right lines, the one on one side the least line GD , and the other on the other side of it.

If therefore any point be taken without a circle, &c. Which was to be demonstrated.

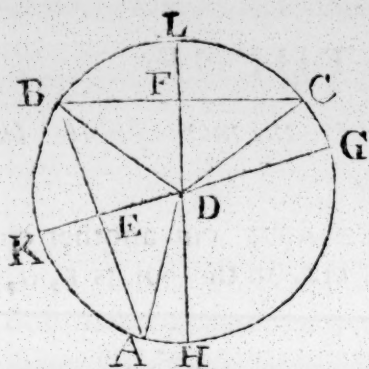
PROP. IX. THEOR.

If some point be taken within a circle, and from it more than two equal right lines be drawn to the circumference; that assumed point is the centre of the circle.

Let the circle be ABC , and the point D taken within it, from which more than two equal right lines drawn to the circumference are equal to one another, viz. the three right lines DA , DB , DC . I say, the point D is the centre of the circle ABC .

For, join AB , BC , and [by 10. 1.] divide them into equal parts, at the points E , F , join ED , DF , which produce to the points G , K , H , L .

Then because AE is equal to EB , and ED is common; the two sides AE , ED will be equal to the two sides BE ,

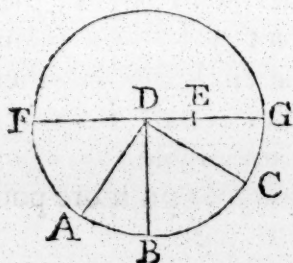


at right angles, the centre of the circle will fall [by cor. 1. 3.] in the bisecting line; the centre of the circle ABC , will fall in the right line GK . For the same reason, the centre of the circle ABC will fall in the right line HL . But the right lines GK , HL have no point but D common to them both. Therefore D is the centre of the circle ABC .

If therefore some point be taken within a circle, and from it more than two equal right lines be drawn to the circumference of the circle; that assumed point is the centre of the circle. Which was to be demonstrated.

Otherwise:

For take some point D , within the circle ABC , and let more than two right lines DA , DB , DC , fall upon the circumference of the circle. I say, the assumed point D , is the centre of the circle ABC .



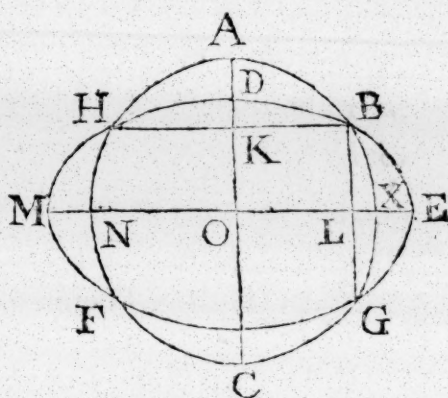
For if not, let, if possible, the point E , be the centre, and having joined DE produce it to FG . Then is FG the diameter of the circle ABC . And so because some point D is taken in the diameter FG of the circle ABC , not being its centre; [by 7. 3.] the greatest line falling upon the circumference will be DG ; but DC is greater than DB , and DB than DA . But they are equal too: which cannot be. Wherefore E is not the centre of the circle ABC . After the same manner we demonstrate that no other point besides D , can be the centre of the circle. Therefore D is the centre of the circle ABC .

PROP.

PROP. X. THEOR.

One circle cannot cut another in more points than two.

For if possible, let one circle ABC cut another circle DEF in more points than two, viz. in the points B, G, H , and join the right lines BG, BH , which bisect in K, L , and draw KC, LM , from K, L at right angles to BG, BH , and produce them to the points A and E .



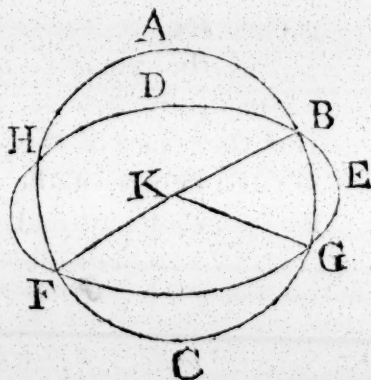
Then because in the circle ABC , the right line AC bisects the right line BH , at right angles; the centre of the circle ABC [by cor. 1. 3.] will fall in AC . Again, because in the same circle ABC , the right line NX cuts the right line BG , into two equal parts, and at right angles too, the centre of the circle ABC will fall in the right line NX . But it has been proved that the centre is in AC too; and the right lines AC, NX , will not meet one another in any other point but O ; therefore O is the centre of the circle ABC . In like manner we prove, that O is the centre of the circle DEF : therefore O is the centre of both the circles ABC, DEF cutting each other, which [by 5. 3.] is impossible.

Therefore one circle cannot cut another in more points than two. Which was to be demonstrated.

Otherwise:

For again, let the circle ABC cut the circle DEF in more points than two, viz. in B, G, F , and let K be the centre of the circle ABC : join KF, KG, KB .

Then because within the circle DEF is taken the point K , from whence more than two equal right lines KB, KF, KG , are drawn to the circumference; the



point K [by 9. 3.] will be the centre of the circle DEF , but K is the centre of the circle ABC . Therefore K will be the centre of two circles that intersect one another. Which [by 5. 3.] is impossible.

Wherefore one circle cannot cut another in more points than two. Which was to be demonstrated.

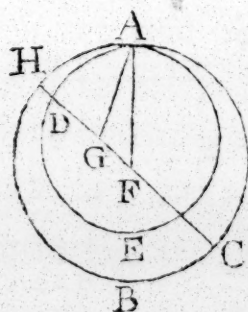
PROP. XI. THEOR.

If two circles touch one another inwardly, and their centres be taken, the right line joining their centres being produced, will fall in the point wherein those circles touch one another.

For let two circles ABC , ADE , touch one another inwardly in the point A , and let F be the centre of the circle ABC , and G the centre of the circle ADE : I say, the right line joining the points G , and F being produced will fall in the point A .

For if not, let, if possible, the right line $FGDH$, joining the centres, fall without the point of contact, and join AF , AG .

Then because AG , GF are [by 20. I.] greater than AF , that is than FH , (for FA is equal to FH , both being drawn from the same centre.) If FG which is common be taken away, the remainder AG is greater than the remainder GH . But AG is equal to GD ; therefore GD is greater than GH , the lesser than the greater: which is impossible. Therefore a right line joining the points F , G , being continued,



will not fall without the point of contact A ; and so necessarily must fall in it.

If therefore two circles touch one another inwardly, and their centres be taken, the right line joining their centres will fall in the point of contact of those circles. Which was to be demonstrated.

Otherwise:

* But let it fall without the point of contact, as GFC , continue out CFG to the point H , and join AG , AF .

* This demonstration scarcely differs from the former, so that it seems not to be Euclid's.

Then

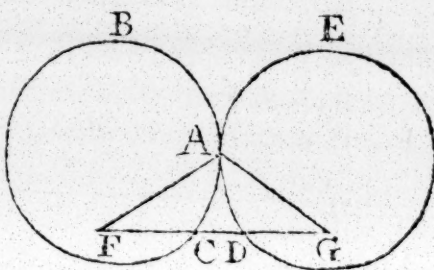
Then because AG , GF are greater than AF , and AF is equal to CF , that is, to FH ; if FG which is common be taken away, the remainder AG will be greater than the remainder GH ; that is, GD is greater than GH , the less than the greater; which cannot be. In like manner we prove the same absurdity will follow, if the centre of the greater circle be without the less.

PROP. XII. THEOR.

If two circles touch one another outwardly, the right line joining their centres will pass thro' the point, wherein the circles touch one another.

For let two circles ABC , ADE , touch one another outwardly in the point A .

Let F be the centre of the circle ABC , and G that of the circle ADE : I say the right line drawn from the point F to G , will pass thro' the point of contact A .



For if not, let it fall without that point, as $FCDG$, and join AF , AG .

Then because F is the centre of the circle ABC , FC will be equal to FA . Again, because G is the centre of the circle ADE , AG will be equal to GD , but it has been proved that FA is equal to FC too. Therefore FA , AG are equal to FC , DG . Consequently the whole FG is greater than FA , AG . But by [20. 1.] it is lesser too; which cannot be. Therefore the right line drawn from F to G , cannot pass otherwise than thro' the point A of contact: wherefore it necessarily passes thro' that point.

If therefore two circles touch one another outwardly, the right line joining their centres, will pass thro' the point of contact of those two circles. Which was to be demonstrated.

PROP. XIII. THEOR.

One circle cannot touch another in more points than one, whether it be inwardly or outwardly.

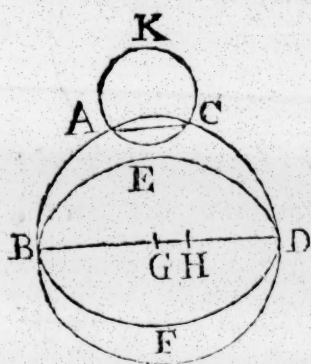
For first, if possible, let the circle $ABDC$, touch the circle $EBFD$ inwardly in more points than one, viz. in

B , D ,

BD , and [by 1. 3.] find G the centre of the circle $ABDC$, and H the centre of the circle $EBFD$.

Then a right line drawn thro' the points G, H , will pass thro' the points B, D . Let this be $BGH D$, and because G is the centre of the circle $ABDC$, BG will be equal to GD .

Therefore BG is greater than HD ; and BH much greater than HD . Again, because H is the centre of the circle $EBFD$, BH is equal to HD ; but it has been proved to be much greater than it; which is impossible. Therefore one circle will not touch another inwardly in more points than one.



Nor secondly, I say one circle will not touch another outwardly in more points than one. For if possible, let the circle ACK touch the circle $ABDC$ outwardly in more points than one, viz. in A, C , and draw AC .

Then because two points A, C , are taken in the circumference of the circles $ABDC, ACK$; the right line joining them [by 2. 3.] falls within them. But [by 3 def. 3] the same line which falls within the circle $ABDC$ will fall without the circle ACK , which is absurd. Therefore one circle cannot touch another outwardly in more points than one; but it has been proved, that one circle cannot touch another inwardly in more points than one.

Therefore one circle cannot touch another in more points than one, whether it be inwardly or outward. Which was to be demonstrated.

PROP. XIV. THEOR.

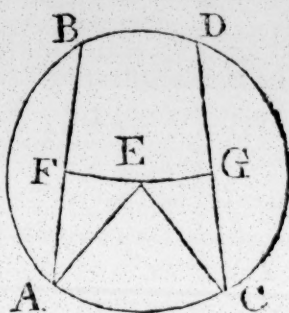
In a circle equal right lines are equally distant from the centre; and those right lines which are equally distant from the centre, are equal to one another.

Let there be a circle $ABDC$, and let the right lines AB, CD in it be equal. I say, they are equally distant from the centre.

For find E the centre of the circle $ABDC$, and from it draw EF, EG perpendicular to AB, CD , and join AE, EC .

Then

Then because the right line EF drawn from the centre cuts the right line AB not drawn from the centre at right angles; [by 3.3.] it will divide it into two equal parts; wherefore AF is equal to FB ; and so AB will be double to AF . Also, for the same reason CD is double to CG ; but AB is equal to CD ; wherefore AF is also equal to CG . And because



AE is equal to EC , the square of AE will be equal to the square of EC . But [by 47. 1.] the squares of AF , FE are equal to the square of AE : For the angle F is a right angle; also the squares of EG , GC are equal to the square of EC , since the angle G is a right one: therefore the squares of AF , FE are equal to the squares of CG , GE : but the square of AF is equal to the square of CG , for AF is equal to CG . Therefore the remaining square of FE will be equal to the remaining square of EG ; and so FE is equal to EG . But in a circle right lines are said [by 4 def. 3.] to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal: therefore AB , CD are equally distant from the centre.

Now let AB , CD be equally distant from the centre, that is, let FE be equal to EG . I say, AB will be equal to CD .

For the construction remaining the same, we demonstrate as above, that AB is double to AF , and CD double to CG : and because AE is equal to EC , the square of AE will be equal to the square of EC . But [by 47. 1.] the squares of EF , FA are equal to the square of AE , and the squares of EG , GC to the square of EC . Therefore the squares of EF , FA are equal to the squares of EG , GC ; but the square of EG is equal to the square of EF , since EF is equal to EG . Therefore the remaining square of AF will be equal to the remaining square of CG ; wherefore AF is equal to CG , but AB is double to AF , and CD double to CG . Therefore AB will be equal to CD .

Wherefore in a circle, equal right lines are equally distant from the centre, and those right lines which are equally distant from the centre are equal to one another. Which was to be demonstrated.

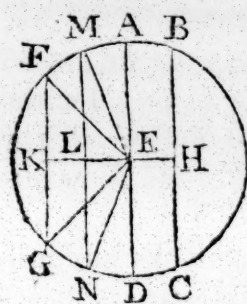
PROP.

PROP. XV. THEOR.

The greatest right line in a circle is a diameter, and of all other right lines in it, that which is nearer to the centre is greater than that which is more remote.

Let $A B C D$ be a circle, whose diameter is $A D$, and centre E , and let $B C$ be nearer to the centre E than $F G$. I say, $A D$ is the greatest right line, and $B C$ is greater than $F G$.

For from the centre E draw the right lines $E H$, $E K$ perpendicular to $B C$, $F G$. Then because $B C$ is nearer to the centre than $F G$ [by 5 def. 3.] $E K$ will be greater than $E H$, make $E L$ equal to $E H$, and through L draw $L M$ at right angles to $E K$; which produce to N , and join $P M$, $E N$, $E F$, $E G$.



Then because $E H$ is equal to $E L$; the line $B C$ [by 14. 3.] will be equal to $M N$. Again, because $A E$ is equal to $E M$, and

$E D$ to $E N$: $A D$ will be equal to $M E$, $E N$; but $M E$, $E N$ are greater than $M N$; therefore $A D$ is greater than $M N$. But $M N$ is equal to $B C$: therefore $A D$ is greater than $B C$. And because the two sides $M E$, $E N$ are equal to the two sides $F E$, $E G$, and the angle $M E N$ is greater than the angle $F E G$; the base $M N$ [by 24. 1.] will be greater than the base $F G$. But $M N$ has been proved to be equal to $B C$: Wherefore the base $B C$ is greater than the base $F G$. Therefore the diameter $A D$ is the greatest of all these right lines, and $B C$ is greater than $F G$.

Wherefore the greatest right line drawn in a circle is a diameter; and of all other right lines in it, that which is nearer to the centre is greater than that which is more remote. Which was to be demonstrated.

PROP. XVI. THEOR.

A right line drawn from the end of the diameter of a circle, at right angles to that diameter, will fall without the circle; and no other right line can be drawn

drawn between that same line and the circumference of the circle; the angle of a semicircle is greater than any right lined acute angle: and the remaining angle (made by that line and the circumference) is less than any right lined angle s.

Let ABC be a circle, whose centre is D , and diameter AB . I say, the right line drawn from the point A , at right angles to AB , will fall without the circle.

For if not, let it fall within if possible, as AC , and join DC .

Then because DA is equal to DC , the angle DAC [by 5. 1.] will be equal to the angle ACD ; but DAC is a right angle; therefore ACD is a right angle too, and so the angles DAC , ACD are equal to two right angles, which [by 17. 1.] cannot be. Therefore a right line drawn from the point A at right angles to BA , does not fall within the circle; after the same manner we demonstrate that it cannot fall in the circumference; therefore it must necessarily fall without the circle, as AE does.

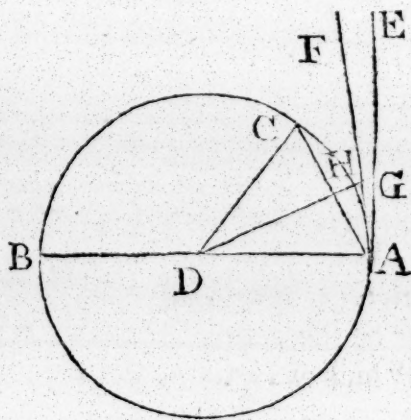
I say again, between the right line AE , and the circumference CHA , no other right line can be drawn.

For if there can, let it be FA , and draw DG from the point D perpendicular to FA .

Then because the angle AGD is a right angle, and DAG is less than a right angle, AD [by 19. 1.] will be greater than DG . But DA is equal to DH : therefore DH is greater than DG , the less than the greater, which is impossible. Therefore no right line can be drawn between AE , and the circumference of the circle.

I say moreover, the angle of the semi-circle, which is contained under the right line BA , and the circumference CHA , is greater than any right lined acute angle, and the remaining angle contained under the circumference CHA , and the right line AE is less than any right lined acute angle.

For



For if any right lined acute angle be greater than the angle contained under the right line BA , and the circumference CHA , or any right lined acute angle be less than that contained under the right line AE , and the circumference CHA , a right line may be drawn between the circumference CHA , and the right line AE making an angle greater than that contained under the right line BA , and the circumference CHA , viz. which is contained under right lines, but less than that contained under the circumference CHA , and the right line AE . But such a right line cannot be drawn; therefore no right lined acute angle is greater than the angle contained under the right line BA , and the circumference CHA , nor is there any one so little as that contained under the right line AE , and the circumference CHA . Which was to be demonstrated.

Cor. From hence it is evident, that a right line drawn from the end of the diameter of a circle at right angles to it, touches the circle, and that in one point only. Because a right line meeting a circle in two points, has been demonstrated [by 2. 3.] to fall within the circle.

§ The two latter parts of this proposition do not appear to be of any great use, in the elements of plain geometry at least; nor has Euclid any where used them in the following part of these elements. If they had been omitted it might perhaps have been better; they have only served as a subject of contention for mathematicians, and caused much vain dispute and ill-natured wrangling amongst them. See the Controversies of Clavius and Peletarius, about the angle of contact. See Dr. Wallis's Mathematicks, Taquet's Euclid, &c.

PROP. XVII. PROBL.

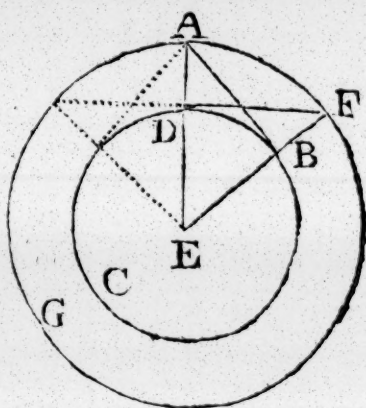
From a given point to draw a right line to touch a given circle^h.

Let the given point be A , and the given circle be BCD . It is required to draw a right line from the point A , to touch the circle BCD .

For find the centre E of the circle; join AE , and about the centre E with the distance EA [by 3. post.] describe the circle AFG ; and [by 11. 1.] draw DF from the point

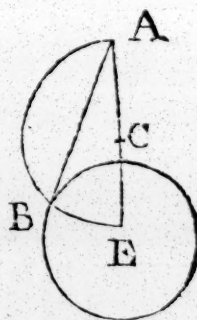
point D at right angles to EA , and join EBF , AB . I say the right line AB is drawn from the point A to touch the circle ABC .

For because E is the centre of the circles BCD , AFG ; EA will be equal to EF , and ED equal to EB : wherefore the two sides AE , EB are equal to the two sides FE , ED , and they contain the common angle at E : therefore the base DF [by 4. 1.] is equal to the base AB , and the triangle DEF equal to the triangle EBA , and the remaining angles equal to the remaining angles: Therefore the angle EBA is equal to the angle EDF . But EDF is a right angle; and so EBA is a right angle too, and EB is a line drawn from the centre. But a right line drawn from the end of the diameter of a circle at right angles to that diameter [by 16. 3.] touches the circle: Therefore AB touches the circle.

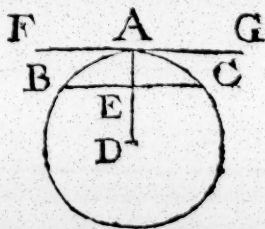


Therefore there is drawn a right line AB from the point A to touch the circle BCD . Which was to be done.

The practice of this problem may be more elegantly effected thus. Join the centre E and the given point A by a right line AE , which bisect in the point C . About C with either of the distances CA , or CE , describe a circle cutting the given circle in the point B . Draw the right line AB , and this will touch the given circle in the point B . But the demonstration depends upon the thirty first proposition of this book.



If it be required to draw a tangent to a circle parallel to a given right line, it may be done thus. Let ABC be a circle, whose centre is D , and let BC be the right line cutting it, to whom a right line is to be drawn parallel to touch the circle. From D draw DE perpendicular to BC , which continue to the point A in the circumference. And from A draw FG perpendicular to AD . Then [by coroll 16.3.] the line FG , which is [by 28. 1.] parallel to BC , touch the circle in the point A .



K

PROP.

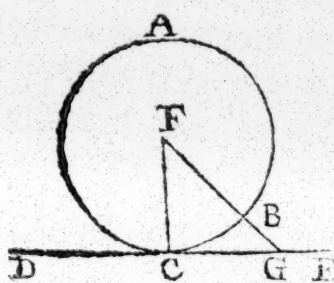
PROP. XVIII. THEOR.

If any right line touches a circle, and a right line be drawn from the centre to the point of contact, this line will be perpendicular to the tangent.

For let a right line DE touch the circle ABC in the point C , and from the centre F of the circle ABC let the right line CF be drawn to C : I say CF is perpendicular to DE .

For if it be not, draw [by 12. 1.] FG from F perpendicular to DE .

Then because the angle FGC is a right angle; the angle GCF [by 17. 1.] will be an acute angle: But [by 19. 1.]



the greater side of a triangle is opposite to the greater angle: therefore FC is greater than FG . But FC is equal to FB . Therefore FB is greater than FG , the less than the greater; which cannot be. Therefore FG is not perpendicular to the right line DE . In the same manner

we demonstrate that no right line whatsoever except FC can be perpendicular to DE : Therefore FC is perpendicular to DE .

If therefore any right line touches a circle, and a right line be drawn from the centre to the point of contact, this will be perpendicular to the tangent. Which was to be demonstrated.

PROP. XIX. THEOR.

If any right line touches a circle, and a right line be drawn from the point of contact at right angles to the tangent; the centre of the circle will be in that same line.*

For let any right line DE touch the circle ABC in C , and let CA be drawn from the point C at right angles to DE : I say the centre of the circle is in the right line AC .

* In the circle.

For if not, let F be the centre, and join CF .

Then because the right line DE touches the circle ABC , and the right line FC is drawn from the centre to the point of contact, the right line FC [by 18. 3.] will be perpendicular to the tangent DE . Therefore

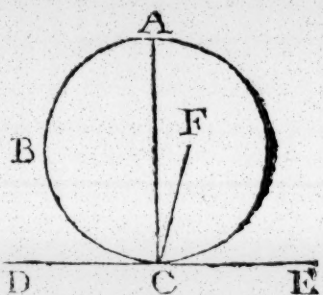
the angle FCE is a right angle. But ACE is a right angle too.

Therefore the angle FCE is equal to the angle ACE , the lesser equal to the greater, which is impossible.

Wherefore F is not the centre of the circle ABC . After this manner

we demonstrate that the centre of the circle is in no other right line soever except AC .

Therefore if any right line touches a circle, and a right line be drawn at right angles to the tangent; the centre of the circle will be in that same right line. Which was to be demonstrated.



PROP. XX. THEOR.

An angle at the centre of a circle, is double to that which is at the circumference, when the same part of the circumference is the base of them both.

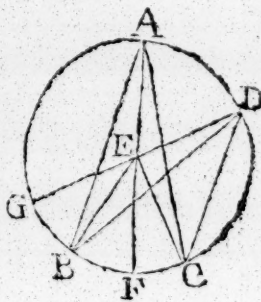
Let ABC be a circle, and let BEC be an angle at its centre, and BAC an angle at its circumference, both having the same part BC of the circumference for a base: I say the angle BEC is double to the angle BAC .

For join AE , which produce to F .

Then because EA is equal to EB ; the angle EAB [by 5. 1.] will be equal to the angle EBA : Therefore the angles EAB, EBA , are double to the angle EAB . But [by 32. 1.] the angle BEF , is equal to the angles EAB, EBA . Therefore the angle BEF is double to the angle EAB .

By the same reason the angle FEC , will be double to the angle EAC . Therefore the whole angle BEC will be double to the whole angle BAC .

Again, Let there be another angle BDC [taken lower] and produce the line DE to G . We demonstrate after the



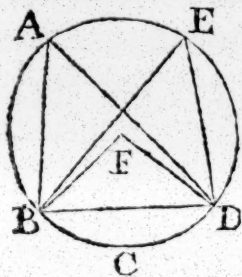
same manner that the angle GEC is double to the angle GDC . But GEB is double to GBD : Therefore the remaining angle BEC is double to the remaining angle BDC .

Therefore an angle at the centre of a circle, is double to that which is at the circumference, when the same part of the circumference is the base of them both. Which was to be demonstrated.

PROP. XXI. THEOR.

All the angles in the same segment of a circle, are equal to one another.

Let there be a circle $ABCD$, and let the angles BAD , BED be in the same segment $BAED$ of that circle: I say these angles are equal to one another.



For [by 1. 3.] find F the centre of the circle $ABCD$, and join BF , FD .

Then because the angle BFD is at the centre, and the angle BAD at the circumference, both having the same part BCD of the circumference for their base: the angle BFD [by 20. 3.] will be double to the angle BAD . For

the same reason, the angle BFD will be double to the angle BED : Therefore the angle BAD [by 7. ax.] will be equal to the angle BED .

Therefore all angles in the same segment of a circle are equal to one another. Which was to be demonstrated.

PROP. XXII. THEOR.

The opposite angles of any quadrilateral figure described in a circle, are equal to two right angles.

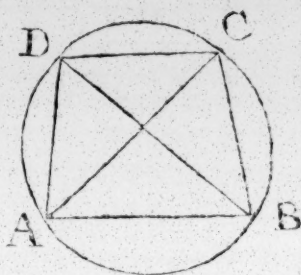
Let there be a circle $ABCD$, and let the quadrilateral figure $ABCD$ be described in it: I say, its opposite angles are equal to two right angles.

Join AC , BD .

Then because [by 32. 1.] the three angles of every triangle are equal to two right angles, the three angles CAB , ABC , BCA of the triangle ABC will be equal to two right angles. But [by 21. 3.] the angle CAB is equal to

to

to the angle BDC ; for they are both in the same segment $BADC$, and the angle ACB is equal to the angle ADB , since they are both in the same segment $ADCB$: Therefore the whole angle ADC is equal to the angles BAC , ACB . Add the common angle ABC to both:



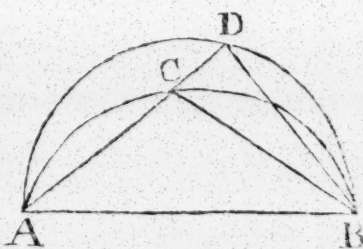
Then the angles ABC , BAC , ACB will be equal to the angles ABC , ADC . But ABC , BAC , ACB , are equal to two right angles: Therefore the angles ABC , ADC also are equal to two right angles. We prove, after the same manner, that the angles BAD , DCB are also equal to two right angles.

Therefore the opposite angles of any quadrilateral figure described in a circle, are equal to two right angles. Which was to be demonstrated.

PROP. XXIII. THEOR.

Two similar and unequal segments of circles cannot be set upon the same right line, on the same side of it.

For if this be possible, let two unequal segments, ACB , ADB , of two circles stand upon the same right line AB , both on the same side of it. Draw ACD , and join CB , DB .



Then because the segment ACB is similar to the segment ADB , and those segments of circles are similar [by II. def. 3.] which receive equal angles; the angle ACB will be equal to the angle ADB , the outward angle equal to the inward one. Which [by I 6. I.] is impossible.

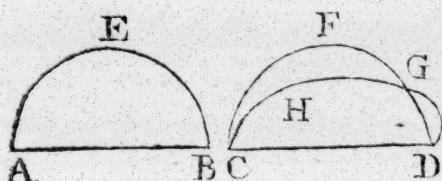
Therefore two similar and unequal segments of two circles cannot be set upon the same right line, on the same side of it. Which was to be demonstrated.

PROP. XXIV. THEOR.

Similar segments of circles, which are upon equal right lines, are equal to one another.

For let AEB , CFD be similar segments of circles standing upon the equal right lines AB , CD : I say the segments AEB , CFD are equal to one another.

For the segment AEB being applied to the segment



CFD so as the point A agrees with C , and the line AB with CD ; the point B will agree with the point D , because AB is equal to CD : But since the right line AB agrees with CD , the segment AEB will also agree with the segment CFD . For if AB agrees with CD , and at the same time the segment AEB should not agree with the segment CFD , but falls into a different situation, as $CHGD$; then as one circle does not cut another [by 10. 3.] in more points than two. But the circle $CHGD$ cuts the circle CFD in more points than two, viz. in the points C , G , D ; which is impossible: Therefore when the right line AB agrees with the right line CD , the segment AEB cannot but agree with the segment CFD , and so will be equal to it.

Therefore similar segments of circles, which are upon equal right lines, are equal to one another. Which was to be demonstrated.

PROP. XXV. PROBL.

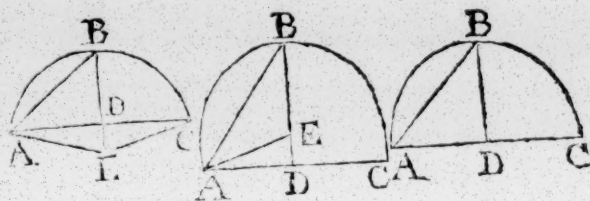
A segment of a circle being given: to describe a circle whose segment it is.

Let ABC be a given segment of a circle: it is required to describe the circle, whose segment ABC is.

Divide [by 10. 1.] AC into two equal parts in D , and [by 11. 1.] draw DB from the point D , at right angles to AC , and join AB . Then the angle ABD is either greater than the angle BAD , or equal to it, or less than it.

First let it be greater. And [by 23. 1.] make the angle BAE at the given point A with the line BA equal to the angle

angle ABD , produce DB to E , and join EC . Then because the angle ABE is equal to the angle BAE ,



the right line BE [by 6. 1.] will be equal to EA . And because AD is equal to DC , and DE is common, the two sides AD , DE are equal to the two sides CD , DE , each to each, and the angle ADE is equal to the angle CDE ; for each of them is a right angle: Therefore [by 4. 1.] the base AE is equal to the base EC . But AE has been proved to be equal to EB : Wherefore BE is also equal to EC , and so the three right lines AE , EB , EC , are equal to one another. Therefore a circle described about the centre E with either of the distances AE , EB , EC will pass through the other points, and [by 9. 3.] will be the circle required to be described. And it is also manifest that the segment ABC is less than a semicircle; because the centre of the circle is without the segment.

But if the angle ABD be equal to the angle BAD ; if AD be made equal to BD or DC , the three right lines DA , DB , DC will be equal to one another; and [by 9. 3.] D will be the centre of the circle to be described; the segment ABC in this case being a semicircle.

Lastly, if the angle ABD be less than the angle BAD , and the angle BAE be made at the given point A , with the right line BA , equal to the angle ABD , the centre E will be in the right line DB , within the segment ABC , which segment will be greater than a semicircle.

Therefore the segment of a circle being given, the circle is described whose segment it is. Which was to be demonstrated.

PROP. XXVI. THEOR.

In equal circles equal angles stand upon equal [parts of their] circumferences, whether they be at their centres or their circumferences.

Let ABC , DEF be equal circles, and let the angles BGC , EHF at their centres be equal, and the angles BAC , EDF at their circumferences be equal: I say the part

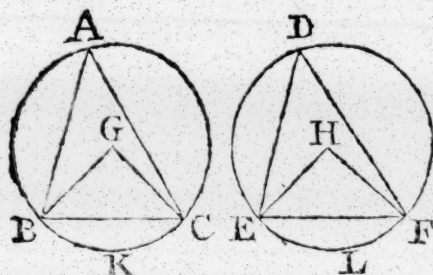
K 4

BGC

BKC of the circumference of one circle, is equal to the part ELF of the circumference of the other.

For join BC, EF .

Then because ABC, DEF are equal circles, the lines drawn from their centres will be equal: Therefore the two sides BG, GC are equal to the two sides EH, HF , and the angle at G is equal to the angle at H : Therefore [by 4. 1]



the base BC is equal to the base EF . And because the angle at A is equal to the angle at D , the segment BAC [by 11. def. 3.] will be similar to the segment EDF , and they are upon the equal right lines BC, EF :

But similar segments of circles which are upon equal right lines [by 24. 3.] are equal to one another: Therefore the segment BAC is equal to the segment EDF . But the whole circle ABC is likewise equal to the whole circle DEF : Therefore the remaining segment BKC is equal to the remaining segment FLF . And so the part BKC of the circumference of one circle will be equal to the part ELF of the circumference of the other.

Therefore in equal circles, equal angles stand upon equal parts of their circumferences, whether they stand at their centres, or at their circumferences. Which was to be demonstrated.

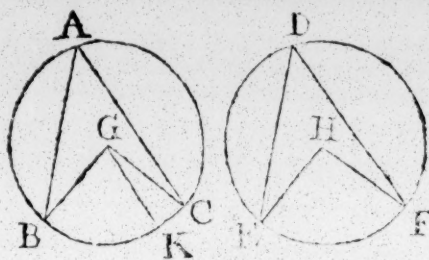
PROP. XXVII. THEOR.

In equal circles, the angles that stand upon equal [parts of their] circumferences, are equal to one another, whether they be at their centres or at their circumferences.

For in the equal circles ABC, DEF , let the angles BGC, EHF at their centres, and the angles BAC, EDF , at their circumferences, stand upon the equal parts BC, EF of the circumferences: I say the angle BGC is equal to the angle EHF , and the angle BAC , equal to the angle EDF .

For if the angle BGC be equal to the equal EHF , it is manifest [by 20. 3.] that the angle BAC is also equal to the angle

angle EDF . But if not, one of them is the greater, which let be BGC , and [by 23. 1.] make the angle BGK , at the point G , with the line BG , equal to the angle EHF . But equal angles [by 26. 3.] stand upon equal parts of the circumferences of the circles when they are at the centres. Therefore the part BK of the one circumference, is equal to the part EF of the other circumference. But the part EF of one circumference is equal to the part BC of the other: Therefore BK is equal to BC , the less to the greater, which cannot be. Therefore the angle BGC is not unequal to the angle EHF : and so they must be equal. But [by 20. 3.] the angle at A is one half the angle BGC , and the angle EDF one half the angle at H : Therefore the angle at A is equal to the angle at D .



Therefore in equal circles, the angles that stand upon equal [parts of their] circumferences, are equal to one another, whether they stand at their centres or circumferences. Which was to be demonstrated.

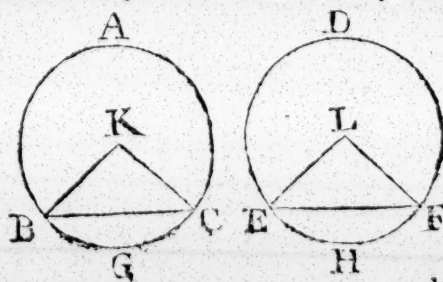
PROP. XXVIII. THEOR.

In equal circles, equal right lines cut off equal parts of their circumferences, the greater part equal to the greater, and the lesser part equal to the lesser.

Let ABC , DEF be equal circles, and let the equal right lines BC , EF cut off the greater parts BAC , EDF , and the lesser parts BGC , EHF of the circumferences: I say the greater part BAC of the one circumference, will be equal to the greater part EDF of the other circumference, and the lesser part BGC equal to the lesser part EHF .

For [by 1. 3.] find the centres K , L of the circles, and join BK , KC , EL , LF .

Then because the circles are equal, the lines drawn from their centres will be equal: Wherefore the two sides BK , KC are equal to the two sides EL , LF , and



the

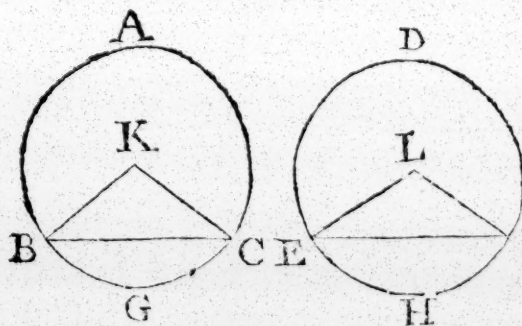
the base BC is equal to the base EF : Therefore the angle BKC [by 8. 1.] is equal to the angle ELF . But [by 26. 3.] equal angles, when they are at the centres, stand upon equal parts of the circumference: Wherefore the part BGC of the one circumference is equal to the part EHF of the other circumference. But the whole circumference ABC is equal to the whole circumference DEF : Therefore the remaining part BAC of the one circumference, will be equal to the remaining part EDF of the other circumference.

Therefore in equal circles, equal right lines cut off equal parts of their circumferences, the greater part equal to the greater, and lesser part equal to the lesser. Which was to be demonstrated.

PROP. XXIX. THEOR.

In equal circles the right lines cutting off equal parts of their circumferences are equal.

Let the circles ABC , DEF be equal, and let BGC , EHF be equal parts of their circumferences; and join BC , EF : I say the right line BC is equal to the right line EF .



For, find K, L the centres of the circles. And join BK , KC , EL , LF .

Then because the part BGC of one circumference is equal to the part EHF of the other; the angle BKC shall be equal to the angle ELF [by 27. 3.]. And because the circles ABC , DEF are equal, the lines drawn from their centres will be equal: Wherefore the two sides BK , KC are equal to the two sides EL , LF , and they contain equal angles: Therefore the base BC [by 4. 1.] is equal to the base EF .

Wherefore in equal circles, the right lines cutting off equal parts of their circumferences are equal. Which was to be demonstrated.

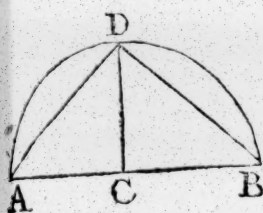
PROP.

PROP. XXX. PROBL.

To divide any given part of the circumference of a circle into two equal parts.

Let $A D B$ be a given part of the circumference of a circle: It is required to divide this given part $A D B$ of the circumference of a circle into two equal parts.

Join $A B$, and [by 10. 1.] bisect the same in C ; from the point C [by 11. 1.] draw $C D$ at right angles to $A B$, and join $A D$, $D B$.



Then because $A C$ is equal to $C B$, and $C D$ is common, the two sides $A C$, $C D$ are equal to the two sides $B C$, $C D$, and the angle $A C D$ is equal to the angle $B C D$, as being each of them a right angle: Therefore the base $A D$ [by 4. 1.] is equal to the base $D B$. But [by 28. 3.] equal right lines cut off equal parts of circumferences, the greater part equal to the greater, and the lesser part equal to the lesser; and each of the parts $A D$, $D B$ of the circumference, is less than half the circumference. Wherefore the part $A D$ of the circumference, will be equal to the part $B D$ of it.

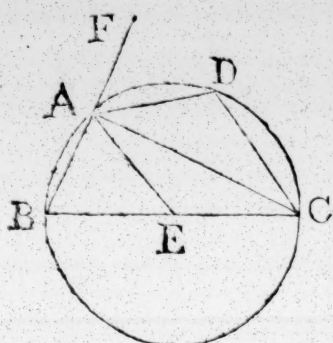
Therefore any given part of the circumference of a circle is divided into two equal parts. Which was to be done.

PROP. XXXI. THEOR.

Any angle in a semicircle is a right angle, any angle in a greater segment is less than a right angle, and any angle in a lesser segment is greater than a right angle; and moreover the angle of a greater segment is greater than a right angle, and the angle of a lesser segment, is less than a right angle.

Let $A B C D$ be a circle, whose diameter is $B C$, and centre E . Take any points A and D , in the circumference, and join $B A$, $A C$, $A D$, $D C$: I say the angle $B A C$ in the semicircle $B A C$ is a right angle: The angle in the segment $A B C$ greater than a semicircle, is less than a right angle, and the angle $A D C$ in the segment $D A C$, which is less than a semicircle, is greater than a right angle.

Join



Join AE , and produce BA to F .
Then because BE is equal to EA ,
the angle EAB is [by 5. 1.] equal
to the angle EBA . Also because
 EA is equal to EC , the angle ACE
will be equal to the angle CAE ;
and so the whole angle BAC , is e-
qual to both the angles ABC , ACB .
But the angle FAC without the tri-

angle ABC , is [by 32. 1.] also equal to both the angles
 ABC , ACB : Wherefore the angle BAC is equal to the
angle FAC : and so each of them [by 10. def. 1.] is a right
angle: Wherefore the angle CAB in a semicircle is a right
angle.

And because the two angles ABC , BAC of the triangle
 ABC [by 17. 1.] are less than two right angles, and BAC
is a right angle; the angle ABC will be less than a right
angle. And it is in the segment ABC , which is greater
than a semicircle.

And since $ABCD$ is a quadrilateral figure in a circle,
and [by 22. 3.] the opposite angles of this figure are equal
to two right angles; the angles ABC , ADC will be equal
to two right angles; and the angle ABC is less than a right
angle: Therefore the remaining angle ADC will be great-
er than a right angle, and it is in the segment ADC , which
is less than a semicircle.

I say moreover the angle of a greater segment contained
by the circumference ABC , and the right line AC , is great-
er than a right angle; and the angle of the lesser segment
contained by the circumference ADC , and the right line
 AC is less than a right angle; which is indeed sufficiently
evident. For because the angle contained by the right lines
 BA , AC is a right angle, the angle contained by the cir-
cumference ABC and the right line AC is greater than a
right angle. Also because the angle contained by the right
lines CA , AF is a right angle: the angle contained by the
right line CA and the circumference ADC , is less than a
right angle.

Therefore any angle which is in a semicircle is a right
angle, any angle in a great segment, is less than a right
angle, and any angle in a lesser segment, is greater than a
right angle; and moreover the angle of a greater segment,

is

is greater than a right angle, and the angle of a lesser segment is less than a right angle. Which was to be demonstrated.

Otherwise :

The angle BAC is proved to be a right angle thus. Because the angle AEC is double to the angle BAE , since [by 32. 1.] it is equal to both the inward opposite angles: and the angle AEB is double to the angle EAC . Therefore the angles AEB , AEC will be double to the angle BAC . But [by 13. 1.] the angles AEB , AEC are equal to two right angles: Therefore the angle BAC will be one right angle. Which was to be demonstrated.

Corollary. From hence it is manifest, If one angle of a triangle be equal to the two others, it is a right angle: because the angle which is adjacent to it, is equal to both those angles; and because [by 10. def. 1.] when adjacent angles are equal, they are each of them a right angle.

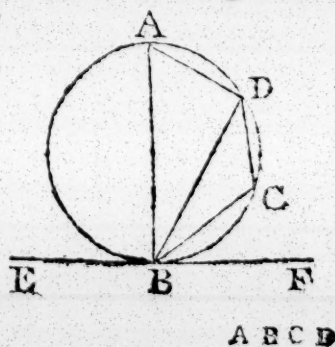
PROP. XXXII. THEOR.

If a right line touches a circle, and a right line be drawn from the point of contact cutting the circle; the angles that this line makes with the tangent, will be equal to the angles that stand in the alternate segments of the circle.

For let the right line EF touch the circle $ABCD$ in B , and draw any right line BD from B cutting the circle: I say the angles that BD makes with the tangent EF , are equal to the angles which stand in the alternate segments of the circle, that is, the angle FBD is equal to any angle in the segment DAB ; and the angle EBD equal to any angle in the segment DCB .

For [by 11. 1.] draw BA from the point B at right angles to EF , take any point C in the circumference BD , and join AD , DC , CB .

Then because the right line EF touches the circle $ABCD$ in the point B , and BA is drawn from the point of contact B at right angles to the tangent, the centre of the circle



Lastly, Let the angle c be obtuse, and at the right line $AB,$

Laſtly, Let the angle c be obtuſe, and at the right line

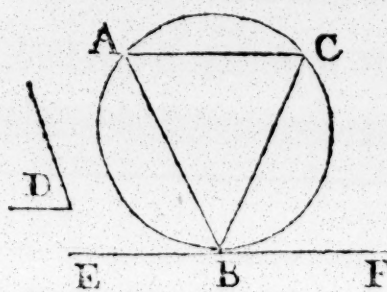
AB, and the point A make [by 23. 1.] the angle BAD, equal to the angle C, as in the third figure; and [by 11. 1.] draw the right line AE at right angles to BD, and again [by 10. 1.] bisect AB in F, draw FG at right angles to AB, and join GB. And because AF is equal to FB, and FG is common, the two sides FG, AF are equal to the two sides BF, FG, and the angle AFG is equal to the angle BFG: Therefore [by 4. 1.] the base AG is equal to the base GB; therefore a circle described with the centre G and distance AG, will pass thro' B. Let it be describ'd as AEB. And because AD is drawn from the end of the diameter AE at right angles to it, the right line AD [by 16. 3.] will touch the circle AEB, and AB is drawn from the point of contact A: wherefore the angle BAD is equal to the angle which is in the alternate segment AHB of the circle. But the angle BAD is equal to the angle C. Therefore the angle C is equal to the angle in the alternate segment AHB of the circle. Wherefore the segment AHB of a circle is described upon the given right line AB, containing an angle equal to the angle C. Which was to be done.

PROP. XXXIV. PROBL.

To cut off a segment from a given circle, containing an angle equal to a given right lined angle.

Let ABC be a given circle, and the given right lined angle D: It is requir'd to cut off a segment from the given circle ABC, containing an angle equal to the given right lin'd angle D.

Draw [by 17. 3.] the right line EF touching the circle



ABC in the point B, and at the right line FE, and point B therein make [by 23. 1.] the angle FBC equal to the angle D.

Then because the right line EF touches the circle ABC, and the right line BC is drawn from the point of contact B; the angle FBC [by 32. 3.] will be equal to that in the alternate segment BAC of the circle. But the angle FBC is equal to the angle D. Therefore the angle in the segment BAC will be also equal to the angle D.

Therefore

Therefore the segment BAC is cut off from the given circle ABC containing an angle equal to the given right lined angle D : Which was to be done.

PROP. XXXV. THEOR.

If two right lines in a circle mutually cut one another, the rectangle contained under the segments of the one, will be equal to the rectangle contained under the segments of the other.

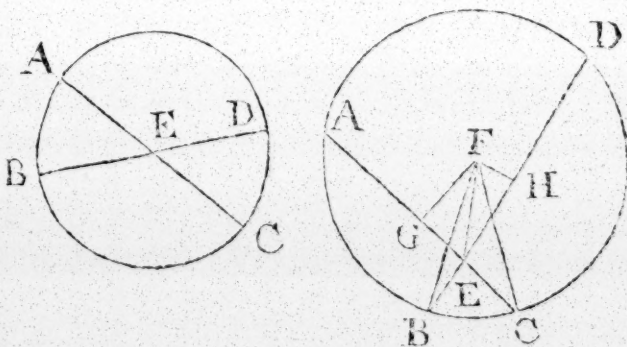
For let two right lines AC, BD in a circle $ABCD$, mutually cut one another in the point E : I say the rectangle contained under AE, EC is equal to the rectangle contained under DE, EB .

Now if AC, BD pass thro' the centre, so that E is the centre of the circle $ABCD$: it is manifest, that AE, EC, DE, EB being then equal, the rectangle contained under AE, EC will be equal to the rectangle contained under DE, EB .

But now let the right lines AC, DB , not pass thro' the centre, and [by 1. 3.] find F the centre of the circle $ABCD$. and [by 12. 1.] from F draw the right lines FG, FH perpendicular to AC, DB , and join FB, FC, FE .

Then because the right line GF drawn thro' the centre, cuts the right line AC

not drawn thro' the centre at right angles; it will [by 3. 3.] bisect the same: so that AG is equal to GC . And because the right line AC is divided into equal parts in the point G , and into unequal ones in the point E , the rectangle contained under AE, EC , together with the square of GE , will [by 5. 2.] be equal to the square of GC . Add the square of GF to both: then the rectangle under AE, EC , together with the squares of GE, GF is equal to the squares of GC, GF . But [by 47. 1.] the square of FE is equal to the squares of EG, GF , and the square of FC equal to the

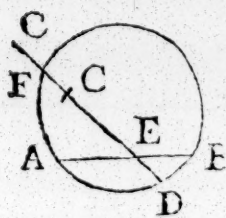


the squares of CG , GF : Therefore the rectangle under AE , EC , together with the square of FE , is equal to the square of FC . But FC is equal to FB : Therefore the rectangle under AE , EC , together with the square of FE , is equal to the square of FB . By the same reason, the rectangle under DE , EB , together with the square of FE , is equal to the square of FB . But it has been proved that the rectangle under AE , EC , together with the square of FE is equal to the square of FB : Therefore the rectangle under AE , EC , together with the square of FE , is equal to the rectangle under DE , EB , together with the square of FE . Take away from both the square of FE ; and then there will remain the rectangle under AE , EC equal to the rectangle under DE , EB .

Therefore if two right lines in a circle mutually cut one another, the rectangle contained under the segments of the one, will be equal to the rectangle contained under the segments of the other. Which was to be demonstrated.

¹ This theorem may be inverted thus: *If two right lines cut one another, that the rectangle contained under the segments of the one, be equal to the rectangle contained under the segments of the other; a circle may be described thro' the extreme points of those two right lines, that is, a circle described thro' any three of those extreme points will also pass thro' the fourth.*

For let the right lines AB , CD mutually cut one another in E , so that the rectangle under AE , EB be equal to the rectangle under CE , ED : I say the four points A , D , B , C do all fall in the circumference of the same circle, and so that circle which passes thro' the three points A , D , B , will necessarily pass thro' the point C . For describe some circle thro' the three points



A , D , B [how this is to be done, see prop. 3. lib. 4.] which if it does not pass thro' C , will pass beyond C , on this side of it, or thro' C . Then because the rectangle under FE , ED is [by 35. 3.] equal to the rectangle under AE , EB , and the rectangle under CE , ED is also put equal to the same rectangle under AE , EB ; the rectangles under CE , ED , and

under FE , ED will be equal. The part and the whole; which is absurd. Therefore the circle passes thro' the point C . Which was to be demonstrated.

This xxxvth proposition, as well as the next, may be more elegantly demonstrated by the sixth book, chiefly from the

proportionality of the sides of equiangular triangles, and by prop. xvi. of that book. But *Euclid* wanted these propositions before, for demonstrating the tenth and eleventh of the fourth book; and therefore he was obliged to demonstrate these propositions in the third book, as well as he could: besides, as the third book treats of the circle, he has put them into their proper place.

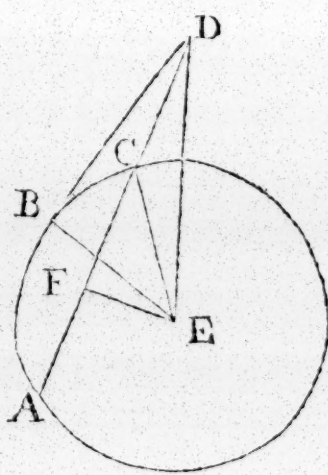
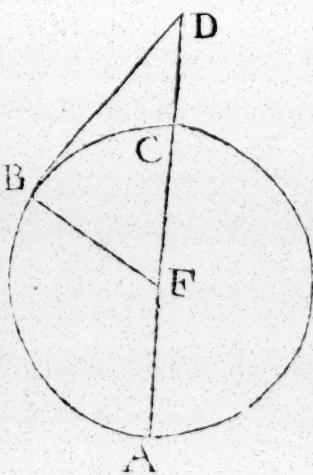
PROP. XXXVI. THEOR.

If any point be taken without a circle, and two right lines be drawn from it to the circle, the one whereof cuts the circle, and the other touches the circle; the rectangle contained under the whole secant [or cutting line] and its outward segment between the assumed point, and the convexity of the circle, will be equal to the square of the tangent.

For let any point *D* be taken without the circle *ABC*, and let two right lines *DCA*, *DB*, be so drawn, that *DCA* cuts the circle, and *DB* touches it: I say the rectangle under *AD*, *DC* is equal to the square of *DB*. Now *DCA* does either pass thro' the centre or not.

First let it pass thro' the centre *F* of the circle *ABC*: and join *FB*. Then [by 18. 3.] the angle *FBD* will be a right angle;

and so because the right line *AC* is bisected in *F*, and the right line *CD* is added to it, the rectangle under *AD*, *DC* together with the square of



FC, will [by 6. 2.] be equal to the square of *FD*; but *FC* is equal to *FB*: Therefore the rectangle under *AD*, *DC*, together with the square of *FB*, is equal to the square of *FD*. But the square of *FD* [by 47. 1.] is equal to the squares of *FB*, *BD*; for the angle *FBD* is a right angle. Therefore the rectangle under *AD*, *DC*, together with the

square of FB is equal to the squares of FB, BD . Take away the square of FB from both, and the rectangle under AD, DC remaining will be equal to the square of the tangent BD .

But now let DCA not pass thro' the centre of the circle ABC , find E the centre, and from E draw [by 12. 1.] EF perpendicular to AC , and join EB, EC, ED ; then is EFD a right angle. And because the right line EF drawn thro' the centre cuts the right line AC not drawn thro' the centre at right angles; it will [by 3. 3.] bisect the same: Wherefore AF is equal to FC . Again, because the right line AC is divided at F into two equal parts, and the right line CD is added to it, the rectangle under AD, DC , together with the square of FC will [by 6. 2.] be equal to the square of FD . Add the square of FE to both: Then the rectangle under AD, DC , together with the squares of CF, FE is equal to the squares of DF, FE . But [by 47. 1.] the square of DE is equal to the squares of DF, FE ; for EFD is a right angle: and the square of CE is equal to the squares of CF, FE : therefore the rectangle under AD, DC , together with the square of CE is equal to the square of ED . But CE is equal to EB ; therefore the rectangle under AD, DC together with the square of EB is equal to the square of ED . But [by 47. 1.] the squares of EB, BD are equal to the square of ED ; since EBD is a right angle: Therefore the rectangle under AD, DC , together with the square of EB , is equal to the squares of EB, BD . Take from both the common square of EB ; and there will remain the rectangle under AD, DC equal to the square of DB .

If therefore any point be taken without a circle, &c. Which was to be demonstrated.

PROP. XXXVII. THEOR.

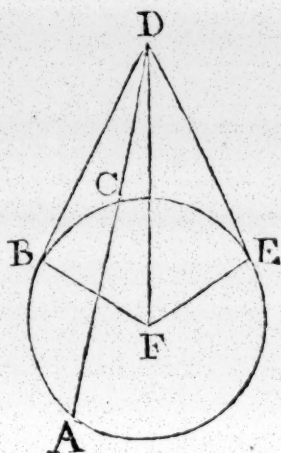
If any point be taken without a circle, and two right lines be drawn from it to the circle, one of which cuts the circle, and the other meets it, and the rectangle contained under the whole secant [or cutting line] and the exterior part of it between the assumed point and the convex part of the circle be equal to the square of the line meeting the circle; this last line will touch the circle.

For

For let any point D be taken without the circle ABC , and let there be two right lines DCA , DB drawn to the circle from the point D , one of which as DCA cuts the circle, and the other, DB meets it, and let the rectangle under AD , DC be equal to the square of DB : I say DB touches the circle ABC .

For [by 17. 3.] draw the tangent DE to the circle ABC , and [by 1. 3.] find the centre F of the circle ABC , and join FE , FB , FD : then [by 18. 3.] the angle FED is a right angle.

And because DE touches the circle ABC , and DCA cuts it, the rectangle under AD , DC will [by 36. 3.] be equal to the square of DE . But the rectangle under AD , DC is put equal to the square of DB : Therefore the square of DE will be equal to the square of DB ; and so the line DE is equal to the line DB : But the line FE is equal to FB too: Therefore the two sides DE , EF are equal to the two sides DB , BF , and they have a common base FD : therefore the angle DEF [by 8. 1.] will be equal to the angle DBF : But DEF is a right angle: wherefore DBF also is a right angle, and FB continued out is a diameter. But a right line drawn from the end of a diameter at right angles to it [by 16. 3.] touches the circle ABC . The demonstration will be the same if the centre of the circle be in AC .



If therefore any point be taken without a circle, &c. Which was to be demonstrated.

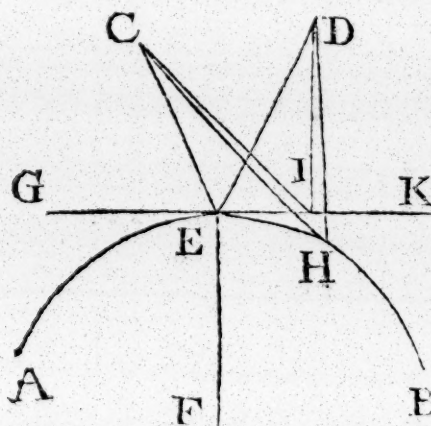
Additions to the Third Book.

PROP. I. THEOR.

If two points be assumed without a given circle, and from the same be drawn two right lines to the convex side of the circumference of the circle; these lines both taken together will be the least when the angles they make with a semidiameter of the circle be equal.

Let there be a given circle AEB , whose semidiameter is the right line FE , and let C, D be two assumed points without the circle, and let two right lines CE, ED be drawn (not cutting thro' the circle) to the point E of the circumference of the circle, making the angles FEC, FED equal to one another: I say these two right lines, taken together, will be less than any other two right lines CH, DH taken together, drawn from the assumed points C, D to any other point H in the circumference of the given circle AEB .

For thro' E [by 11. 1.] draw the right line GK perpendicular to the semidiameter FE . Then [by cor. 16. 3.] GK will touch the circle in the point E . And since the right angles FEG, FEK are equal, and the angles FEC, FED are also equal [by supposition] therefore the angles GEC, DEK will be equal to one another. And because [by 16. 3.] the tangent GK falls without the



circle, the tangent GK will cut the right lines CH, DH without the circle. Take any point I upon the tangent within the angle CHD , and join the right lines CI, DI . Now because the angles GEC, DEK have been proved to be

be equal to one another; [by 1. of addit. to lib. 1.] CE, ED together will be less than CI, ID together, but CI, ID together [by 21. 1.] are less than CH, DH together. Therefore CE, ED together will be much less than CH, DH together. Therefore &c. Which was to be demonstrated.

SCHOLIUM.

If it were required to find a point in the circumference of a given circle such, that drawing right lines from two given points without the circle to it, the angles made by these lines with a semidiameter, or with a tangent to the circle drawn thro' that point, shall be equal: The construction of such a problem generally could not be obtained by the postulations of Euclid, that is, by means of a right line and a circle.

PROP. II. THEOR.

In a circle any two parallel right lines intercept equal parts of the circumference.

Let $ABDC$ be a given circle, and AB, CD any two parallel right lines drawn in it, intercepting the parts AC, BD of the circumference: I say these arches or parts AC, BD of the circumference will be equal to one another.

For join the points A, D , by the right line AD . Then because the right lines AB, CD are parallel, the alternate angles BAD and ADC will [by 29. 1.] be equal to one another. Therefore [by 26. 3.] the parts AC, BD of the circumference of the circle will be equal.

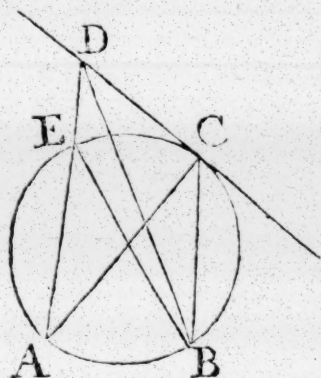
Wherefore, &c. Which was to be demonstrated.



PROP. III. THEOR.

If from any two points of the circumference of a given circle to any point of an assumed tangent of that circle be drawn two right lines, the angle that these right lines make with one another at the tangent will be the greatest of all when the assumed point is that of contact.

Let A, B be two given points in the circumference of the given circle $AECB$; and let the right line DC touch the circle $AECB$ in the point C ; draw the right lines AC, BC to the point of contact C , and to any other point D in the tangent from the given points A, B draw the right lines AD, BD : I say the angle ACB will be greater than the angle ADB .



For let AD cut the circumference of the given circle in the point E , and join the right line BE .

Then because the angles AEB, ACB in the same segment are [by 21. 3.] equal to one another. But the angle AEB is [by 21. 1.] greater than the angle ADB . Therefore will the angle ACB be

greater than the angle ADB .

Therefore, &c. Which was to be demonstrated.

PROP. IV. PROBL.

Any right line being given, and two points over it; it is required to find a point in that given right line such, that drawing two right lines from the given points to it, the angle contained by them shall be greater than any other angle contained under two right lines drawn from those given points to any other point in the given right line.

Let the given right line be CD , and the given points over it A, B : it is required to draw two right lines from the points A, B , to the right line CD , such, that the angle contained under them shall be greater than any other angle contained under two right lines drawn from the points A, B to any other point in the right line CD .

First let the given points A, B be so situate as not to be in a right line parallel to the given right line CD . Join the points A, B by a right line, which being continued, shall cut the right line CD in the point D . Make [by 14. 2.] DC such, that its square shall be equal to the rectangle under AD, DB ; then will C be the point required. For if two right lines AC, BC be drawn to it, the angle ACB

will

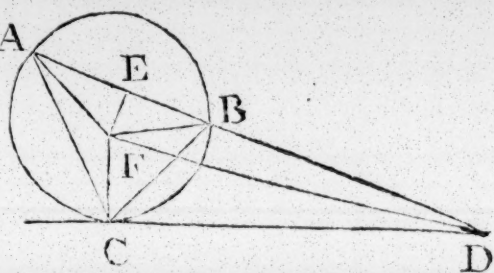
will be the greatest of all such angles at the right line CD .

For bisect AB in the point E , and from E draw EF perpendicular to AB . Also from the point C erect CF perpendicular to CD meeting EF in the point F .

Join AF, FB, FD .

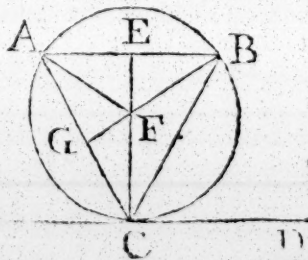
Then because [by construction] the square of CD is equal to the

rectangle under AD, BD . But since [by construction] AB is bisected in E , and so [by 6. 2.] the rectangle under AD, BE together with the square of EB is equal to the square of ED . Therefore will the squares of CD, EB be equal to the square of ED . But since FEB is a right angle, and so [by 47. 1.] the squares of EF, EB are equal to the square of FB , the squares of CD, FB will be equal to the squares of ED, EF . And because FCD, FED are each right angles [by 47. 1.] the squares of CD, CF will be equal to the squares of ED, EF , each of these squares together being equal to the square of FD . Wherefore the squares of CD, FB will be equal to the squares of CD, FC , and so CF, FB will be equal. And since FB is equal to AF [by construction, and by 8. 1.] Therefore a circle ACB described about the centre F with the distance FC will pass thro' the points A, B . And so because the square of CD is [by construction] equal to the rectangle under AD, DB ; the right line CD will [by 38. 3.] touch the circle ACB in C . Therefore [by 3. of this] the angle ACB will be the greatest of all, when its point is at the point of contact C .



Secondly, Let the right line AB [fig. 2.] joining the given points A, B be parallel to the given right line CD . Bisect AB in E , as before; and from E draw the right line EC perpendicular to AB , cutting the given right line CD in the point C . Then if AC, BC be joined, I say the angle ACB will be greater than any other angle terminating in the right line CD , and whose sides pass thro' the given points A, B .

Fig. 2.



For

For bisect AC in the point G , and from G draw the right line GF perpendicular to AC , meeting the right line EC in F , and join FB .

Then [by construction and 8. 1.] AF will be equal to FB , and so will FC be equal to either of them. Therefore a circle ABC described about the centre F with either of the distances FC , FA , FB will pass thro' the other two points. And since ECD is a right angle [by construction] therefore [by 16. 3.] the right line CD will touch the circle ABC in the point C . Wherefore [by 3. of this] the angle ACB will be greater than any other angle contained under two right lines drawn from the given points A, B to any other point in the right line CD .

Wherefore, &c. Which was to be demonstrated.

PROP. V. THEOR.

If the sides, or their continuations, of two triangles, having the base of the segment of a circle for their base, and both being on the same side, intercept equal parts of the circumference of the circle, the angles contained under their sides, or their continuations, will be equal.

For let $ACDB$ be a given circle, and AB the base of the segment ACB . Let AGB, AHB be two triangles standing on the same side upon the base AB of the segment, the sides AG, BG of the one, or their continuations [see fig. 2.] intercepting the arch CD equal to the arch EF intercepted by the sides AH, BH , or their continuation of the other. I say the angles AGB, AHB will be equal to one another.

For join the points $A, D; B, E$.

Then because [by 21. 3.] the angle ADB [fig. 1.] is equal to the angle AEB , and the angle CAD is equal to FBE , since the arches CD, EF [by supposition] are equal. And because the outward angle of every triangle [by 32. 1.] is equal to both the inward and opposite angles, the angles GAD, AGD will be equal to the angles EBH, AHB . And taking away the equal angles GAD, EBH there will remain the angles AGB, AHB equal to one another.

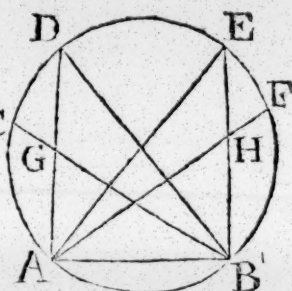
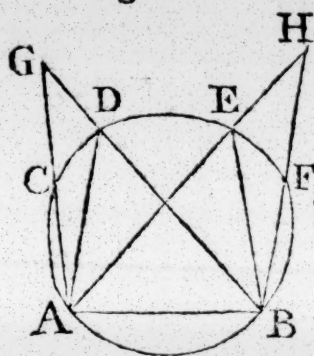
Again

Again [fig. 2.] since the angle $A D B$ [by 21. 3.] is equal to the angle

Fig. 1.

Fig. 2.

$A E B$, and the angle $D B G$ is equal to the angle $E A H$, the arches $C D$, $E F$ being equal; therefore the angles $A D B$, $D B G$ will be



equal to the angles $A E B$, $E A H$; and so the outward angles $A G B$, $A H B$ of the triangles $G D B$, $A E H$ will be [by 32. 1.] equal to one another.

Wherefore, &c. Which was to be demonstrated.

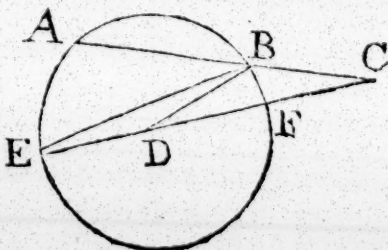
PROP. VI. THEOR.

If any right line be taken in a circle, and the same be continued out till the external part be equal to the semidiameter of the circle, and if a right line be drawn from the farthest extremity of this part thro' the centre of the circle to its concavity, the two parts of the circumference intercepted between these right lines will the one be a third part of the other.

Let $A B F E$ be a circle whose centre is D , and $A B$ be any right line taken in it, which let be continued out till the external part $B C$ be equal to the semidiameter $B D$ of the circle; and from the point C , let the right line $C E$ be drawn thro' the centre D , cutting the circumference of the circle in the points E , F : I say the arch $B F$ intercepted by the right lines $A C$, $E C$ will be one third part of the arch $A E$.

For join the right line $E B$.

Then because [by supposition] the right lines $B C$, $B D$ are equal to one another, the two angles $B C D$, $B D C$ [by 5. 1.] will be equal to one another. In like manner since $E D$, $D B$ are equal



to

to one another; the angles DEB , EBD will be equal to one another; and so [by 32. 1.] the external angle BDC will be double to the angle DEB ; that is, the angle BCD will be double to BEC , since the angles BCD , BDC have been proved to be equal to one another. Therefore those angles taken together, or the external angle ABE will be thrice the angle BEF ; and so since [by 26. 3.] equal parts of the circumference of the circle are intersected by the sides of equal angles: therefore the arch AE will be thrice the arch BF .

Therefore, Q.E.D. Which was to be demonstrated.

SCHOLIUM.

The solution of the following problem is easily obtained from this theorem, viz. An arch of a circle being given, to find an arch that shall be thrice thereof. But the contrary problem, to find an arch of a circle which shall be one third part of a given arch, cannot be solved by Euclid's postulates, viz. a right line and a circle.

PROP. VII. THEOR.

If any two right lines within a circle mutually intersect one another at right angles; the sums of each two of the opposite arches between the extremities of those right lines will be equal to one another.

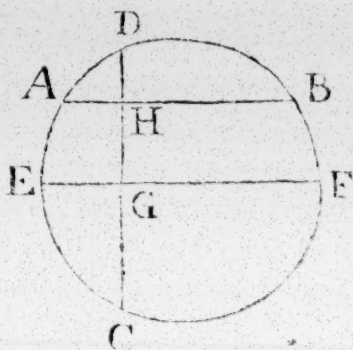
Let ABC be a circle wherein the two right lines AB , CD intersect one another at right angles at the point H : I say the sum of the opposite arches AD , CB will be equal to the sum of the opposite arches AC , DB .

For assume the centre of the circle [by 1. 3.] and thro' it [by 31. 1.] draw the diameter EF parallel to the right line AB .

Then because AB is perpendicular to CD , and EF is parallel to AB ; the angle DGF [by 29. 1.] will be a right angle; and so [by 3. 3.] the right line CD will be bisected in the point G . Wherefore [by 8. 1.] the right lines joining ED , EC will be equal to one another. And so [by 28. 3.] the arches EC , ED will be equal to one another. In like manner CF , DF will be equal. And since the arches EDF , ECF are also equal to one another,

as

as being each of them one half the circumference of the circle, and the arch ED is equal to the arches EA, AD ; the arch CF together with the two arches EA, AD will be equal to one half the circumference. But the arch EA [by 2. of this] is equal to the arch BF . Therefore the arch CB together with the arch AD ,



will be equal to one half the circumference, and the two remaining arches EC, EA , viz. the arch AC together with the arch DB will be equal to one half the circumference too. Therefore the opposite arches AD, CB and AC, DB will be equal to one another.

Wherefore, &c. Which was to be demonstrated.

SCHOLIUM.

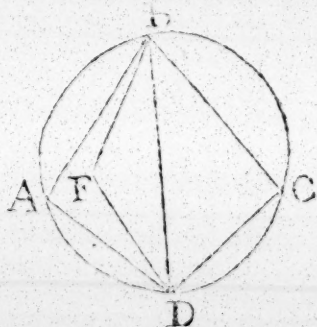
There is another theorem of this kind, whose demonstration, which is easy, I shall omit, viz. That if the continuations of two right lines in a circle cut one another at right angles without the circle, the difference between twice the least of the opposite intercepted arches and the greatest, will be equal to one half the sum of the other two intercepted opposite arches.

PROP. VIII. THEOR.

If the opposite angles of any quadrilateral figure be equal to two right angles, that circle which passes thro' any three of the angles of the figure will also pass thro' the fourth.

Let $ABCD$ be a quadrilateral figure, whose opposite angles A, C , and D, B are equal to two right angles: I say a circle $BADC$ passing thro' the three angles B, C, D of the figure will also pass thro' the fourth angular point A .

For if the circle $ABCD$ does not pass thro' the fourth angular point A : Let it pass, if possible, thro' some other point F .



Draw

Draw the right lines FB , FD , BD .

Then [by 22. 3.] the angles C , F are equal to two right angles. But [by supposition] the angles C and A are also equal to two right angles. Therefore the angles A , F will be equal to one another: Which cannot be, since [by 21. 1.] the angle F is greater than A .

Therefore, &c. Which was to be demonstrated.

PROP. IX. THEOR.

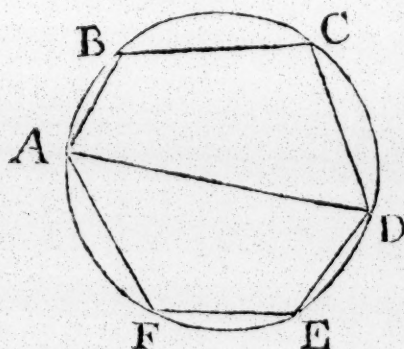
If there be any right lined figure having an even number of sides in a circle, the sum of all the angles of the figure beginning at any one, succeeding one another according to the odd numbers, will be equal to the sum of all the angles, succeeding one another according to the even numbers.

Let $ABCDEF$ [fig. 1.] be a figure in a circle having six sides: I say the sum of its three angles A , C , E will be equal to the sum of its three angles B , D , F .

For join the first angle A , and the fourth D , by a right line.

Then because [by 22. 3.] the opposite angles ADC ,

Fig. 1.



ABC of the quadrilateral figure $ABCD$ are equal to the opposites angles BAD , BCD ; also the opposites angles ADE , AFE of the quadrilateral figure $ADEF$ equal to the opposite angles FAD , FED . Therefore the sums of the angles ADC , ADE , ABC , AFE will be equal to the sum of the angles BAD , FAD , BCD , FED . But the angles ADC ,

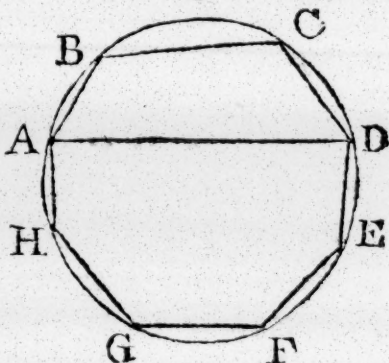
ADE are equal to the angle CDE , and the angles BAD , FAD equal to the angle BAF . Therefore the sum of the three angles BAF , BCD , FED will be equal to the sum of the three angles ABC , CDE , AFE .

Again, Let the figure $ABCDEFGH$ [fig. 2.] in the circle have eight sides: I say its four angles A , C , E , G will be equal to its four angles B , D , F , H .

For divide the figure into the figure $ABCD$ having four sides, and the figure $AHGFED$ having six sides, by draw the right line AD from the first angle A to the fourth D .

Then [by 22. 3.] the angles BAD , C will be equal to the angles CDA , B , and the angles HAD , E , G [by what has been already demonstrated] equal to the angles ADE , F , H . Therefore [by adding] the angles BAD , HAD , C , E , G will be equal to the angles CDA , ADE , B , F , H : that is, since BAD , HAD are equal to A , and ADE , CDA equal to D ; the angles A , C , E , G will be equal to the angles B , D , F , H .

Fig. 2.

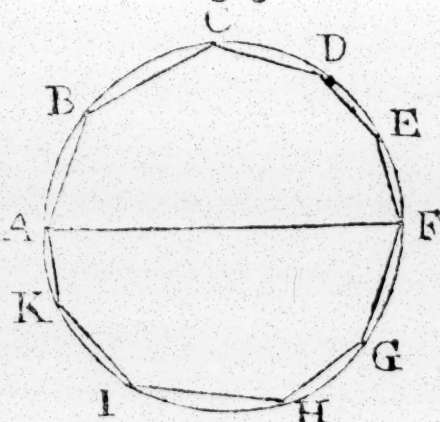


Again, Let the figure $ABCDEFGHIK$ [fig. 3.] have ten sides: I say its five angles A , C , E , G , I will be equal to its five angles B , D , F , H , K .

For divide the figure into two other figures having six sides, by the right line AF , drawn from the first angle A to the sixth angle F .

Then [by the last case but one] the angles BAF , C , E will be equal to the angles B , D , EFA ; and the angles KAF , G , I equal to the angles AFG , H , K .

Fig. 3.



Therefore [by adding] the angles BAF , KAF , C , E , G , I will be equal to the angles B , D , EFA , AFG , H , K ; that is because the angles BAF , KAF are equal to the angle A , and the angles EFA , AFG equal to the angle F , the five angles A , C , E , G , I will be equal to the five angles B , D , F , H , K .

After the same manner the theorem is demonstrated, when the number of even sides is twelve, fourteen, &c.

Therefore, &c. Which was to be demonstrated.

SCHO-

SCHOLIUM.

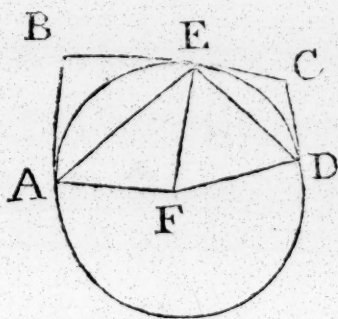
Hence it appears, that the xxii^d proposition of Euclid's third book is only a particular and the most simple case of this general proposition; being that where the right lined figure in the circle has but four sides; also it is easy to demonstrate that a circle passing thro' any three angles of a right lined figure of an even number of sides, whose angles have the conditions as mentioned in this our proposition, will pass thro' the other angles of the right-lined figure.

PROP. X. THEOR.

In a circle if there be three tangents meeting one another, and from the point of contact of the middle tangent be drawn two right lines to the points of contact of the other two tangents, the angles contained under the tangents taken together will be double to the angle contained under the right lines joining the point of contact.

Let the three right lines AB , BC , CD touch the circle AED in the points A , E , D , and let the middle tangent BC meet the other two in the points B , C : I say the angles ABE , ECD , taken together, will be double to the angle AED .

For [by 1. 3.] find F the centre of the circle, and draw the semidiameters FA , FE , FD to the points of contact.



Then because [by 5. 1.] the angles FAE , FEA are equal to one another, as also FDE , FED . Therefore [by 32. 1.] twice the angle FAE together with the angle AFE will be equal to two right angles. In like manner twice the angle FED , together with the angle FED will be equal to two right

angles. And so [by adding] twice the angle FAE , twice the angle FED , together with the angles AFE , EFD will be equal to four right angles. But because the semidiameters FA , FE , FD [by 18. 3.] are at right angles to the tangents AB , BC , DC , and since all the angles of any quadrilateral figure are equal to four right angles; therefore the

the angles $A F E$, $A B E$ of the quadrilateral figure $A B E F$, will be equal to two right angles; in like manner the angles $D F E$, $D C E$ of the quadrilateral figure $F E C D$ will be equal to two right angles: Wherefore [by adding] the angles $A F E$, $A B E$, $D F E$, $D C E$ will be equal to four right angles. Therefore twice the angle $F A E$, twice the angle $F E D$, together with the angles $A F E$, $D E F$, will be equal to the angles $A E F$, $A B E$, $D F E$, $D C E$; and taking away the angles $A F E$, $D F E$, which are common, there will remain twice the angle $F A E$, together with twice the angle $F E D$, equal to the angles $A B E$, $D C E$, that is, since the angle $A E D$ is equal to the angles $F A E$, $F E D$, twice the angle $A E D$ will be equal to the angles $A B E$, $D C E$.

Therefore, *&c.* Which was to be demonstrated.

Corollary. Hence if each side of a right lined figure touches a circle, that is, if a right lined figure circumscribes a circle, and right lines be orderly drawn joining the points of contact forming a right lined figure of the same number of sides within the circle, any two adjacent angles of the circumscribing figure will be double to the angle of the figure within the circle, which is between them.

P R O P. XI.

If each side of any right lined figure, whose sides are even in number, touches a circle, the sum of the first, third, fifth, &c. beginning at any one side, and orderly proceeding, according to the odd numbers, will be equal to the sum of the second, fourth, sixth, &c. sides, orderly proceeding according to the even numbers.

First, let the right lined figure [fig. 1.] have four sides $A B$, $B C$, $A D$, $D C$, and let these sides touch the circle $E F G H$, in the points E , F , G , H ; I say the sum of the first side $A B$, (making $A B$ the first, and $B C$ the second) and the third side $C D$ will be equal to the sum of the second side $B C$, and the fourth side $A D$.

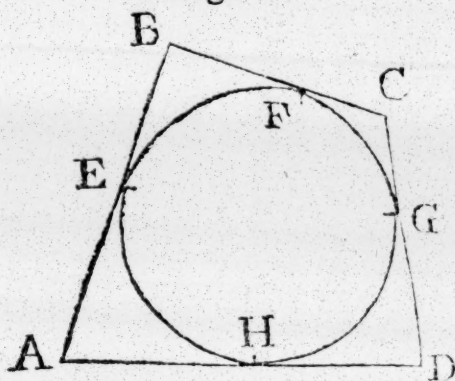
For since $A E$ touches the circle in E , the rectangle under any right line drawn from A to the convex part of the circle, and its external part will [by 37. 3.] be equal to the square of the tangent $A E$: So also because $A H$ touches the circle in H , the rectangle under any other

M

right

right line drawn from A, to the convexity of the circle and its external part, will be equal to the square of the tangent A H; wherefore the squares of the tangents A E, A H will be equal to one another; and so their sides A E, A H will be equal to one another. In like manner it is proved

Fig. 1.

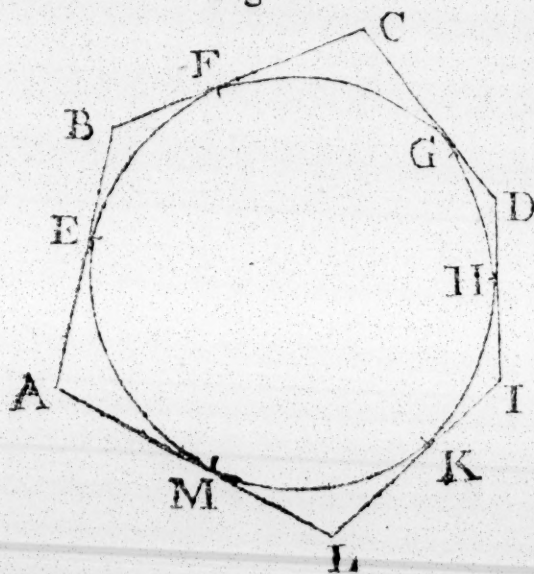


that E B will be equal to B F; F C equal to C G; and G D equal to D H: Therefore A E, B F will be equal to A B; E B, F C equal to B C; F C, G D equal to C D; and D G, A H equal to A D. Wherefore [by adding] A E, B F, F C, G D will be equal to A B, C D. But since B C, A D are equal to E B, F C, G D, A H, that is (because E B has been proved to be equal to B F, and H D equal to D G) equal to A E, B F, F C, G D; therefore will A B, C D be equal to B C, A D.

Again, Let the figure [fig. 2.] have six sides, A B the first, B C the second, C D, D I, I L, A L touching the circle in the points E, F, G, H, K, M: I say A B, C D, I L will be equal to B C, D I, A L.

For from what has been already proved above, it follows that the several tangents A E, A M; B E, B F; C F, C G; D G, D H; H I, I K; K L, L M; will be each two equal to one another. There-

Fig. 2.



fore A B will be equal to A E, B F; B C equal to B E, C F; C D equal to C F, G D; D I equal to G D, H I; I L equal to H I, K L; and A L equal to K L, A M: Therefore [by adding] A B, C D, I L will be equal to A E, B F, C F, G D, H I, K L; and B C, D I, A L equal to B E, C F, G D, H I, K L, A M; and [adding equals to equals] A B, C D, I L,

B E,

BE, CF, GD, HI, LK, AM will be equal to BC, DI, AL, AE, BF, CF, GD, HI, LK; and taking away from both CF, GD, HI, LK, there will remain AE, CD, IL, BE, AM equal to BC, DI, AL, AE, BF. But since BE, AM are equal to AE, BF, therefore again taking these away from both, there will remain AB, CD, IL equal to BC, DI, AL.

The demonstration is much the same, when the figure about the circle has eight, ten, &c. sides.

Therefore, &c. Which was to be demonstrated.

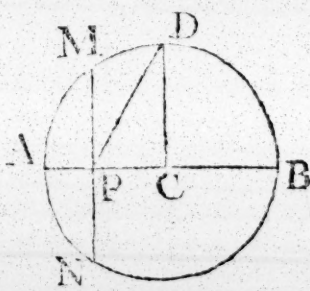
PROP. XII.

If from any point in the diameter of a circle be drawn two right lines to the circumference, the one perpendicular to that diameter, and the other to the middle point of the circumference of one of the semicircles upon that diameter, the squares of these two right lines taken together, will be equal to twice the square of the semidiameter of the circle.

Let ADBN be a circle whose diameter is AB, and centre C, and let D be the middle point of the circumference ADB of the semicircle ADB; also let P be any point in the diameter AB, from which are drawn to the circumference the right line PM perpendicular to AB, and the right line PD to the point D: I say the squares of PM, PD together, will be equal to twice the square of the semidiameter AC of the circle.

For continue out MP meeting the circumference in the point N, and join CD.

Then because MP is perpendicular to the diameter AB; [by 3. 3.] MP, PN will be equal to one another: And therefore [by 35. 3.] the square of PM will be equal to the rectangle under AP, PB. But since [by 5. 2.] the rectangle under AP, PB together with the square of PC is equal to the square of AC; therefore will the squares of PM, PC be equal to the square of AC. But because of the right angle PCD (since the arches AD, DB are [by supposition] equal to one another, and so [by



M 2

27.

27. 3.] the angles ACD, DCB too) the square of PD will be [by 47. 1.] equal to the squares of PC, CD ; that is, because CD, AC are equal, the square of PD will be equal to the squares of PC and AC . Therefore [by adding equals to equals] the squares of PM, PC, PD will be equal to twice the square of AC together with the square of PC ; and taking away from both the common square of PC , there will remain the squares of PM, PD equal to twice the square of AC .

Therefore, Q.E.D. Which was to be demonstrated.

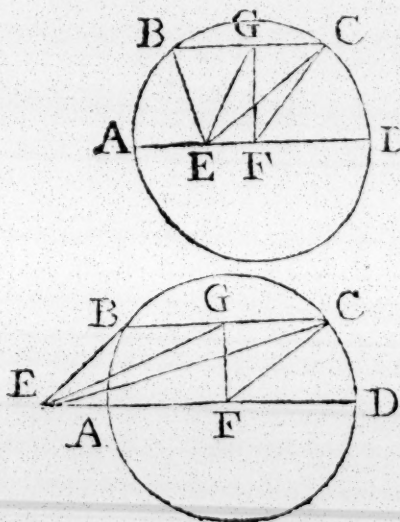
PROP. XIII. THEOR.

In a circle if any right line be parallel to a diameter, and from the extremes of that parallel to any point in that diameter be drawn two right lines. Their squares taken together, will be equal to the squares of the parts of the diameter taken together one each side the assumed point.

Let $ABCD$ be a circle whose diameter is AD , and centre F ; let BC be any right line in the circle parallel to the diameter AD , and let E be any point taken in that diameter, from which let two right lines EB, EC be drawn to the extremities of that parallel: I say the squares of BE, CE taken together will be equal to the squares of the parts AE, ED of the diameter.

For bisect BC [by 10. 1.] in the point G , and join EG, FG, FC .

Then because the side BC of the triangle BEC is bisected in G ; [by prop. 3. add. 2.] the squares of EB, EC will be equal to twice the square of GC , together with twice the square of EG . But because BC [by supposition] is parallel to AD ; the angles AFG, FGC [by 29. 1.] will be equal to one another. But since BC is bisected in G , the angle FGC [by 3. 3.] will be a right angle; therefore AFG will also be a right angle: And so



[by 47. 1.] the square of EG will be equal to the squares of EF , FG ; and twice the square of EG will be equal to twice the squares of EF , FG . Again, since FGC is a right angle, and so the squares of FG , GC are [by 47. 1.] equal to the square of FC , or to the square of AF (for AF is equal to FC), and twice the square of FG , and twice the square of GC is equal to twice the square of AF . Therefore, if equals be added to equals, twice the square of EG , twice the square of FG , together with twice the square of GC , will be equal to twice the square of EF , twice the square of FG , together with twice the square of AF . And taking away from both twice the square of FG ; there will remain twice the square of EG , together with twice the square of GC , equal to twice the square of EF , together with twice the square of AF . But it has been already proved that twice the square of GC , together with twice the square of EG is equal to the squares of EB , EC : Therefore the squares of EB , EC will be equal to twice the square of EF , and twice the square of AF ; that is, because AD is bisected in F ; and so [by 9. and 10. 2.] the squares of AE , ED are equal to twice the squares of EF , AF ; the squares of EB , EC will be equal to the squares of AE , ED .

Therefore, $\mathcal{E}c$. Which was to be demonstrated.

PROP. XIV. THEOR.

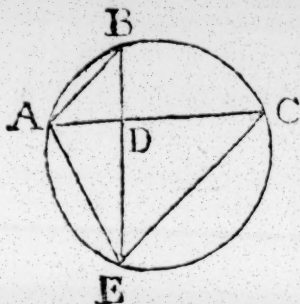
In a circle if any two right lines mutually cut one another at right angles; the four squares of their segments will be equal to the square of the diameter of the circle.

In the circle $ABCE$ let any two right lines AC , EB mutually cut one another at right angles in the point D : I say the four squares of the segments AD , DB , ED , DC will be equal to the square of the diameter of the circle.

For join AB , EC , AE .

Then because ADE [by supposition] is a right angle, and so [by 32. 1.] the angles AED , EAD together are equal to one right angle, the arches AB , EC , on which those angles stand, will be [by 26. 3.] together equal to

one half the circumference of the circle. Therefore be-



cause the angle of a semicircle is [by 31. 3.] a right angle, the squares of AB , EC [by 47. 1.] will be equal to the square of the diameter of the circle. And because the angles at D are right angles [by supposition]; the squares of AD , DB will be equal to the square of AB , and the squares of ED , DC equal

to the square of EC . Therefore [by adding equals to equals] the squares of AD , DB , DE , DC will be equal to the squares of AB , EC . But the squares of AB , EC have been proved to be equal to the square of the diameter. Therefore the squares of AD , BD , ED , CD will be equal to the square of the diameter of the circle $ABCE$.

Therefore, &c. Which was to be demonstrated.

PROP. XV. THEOR.

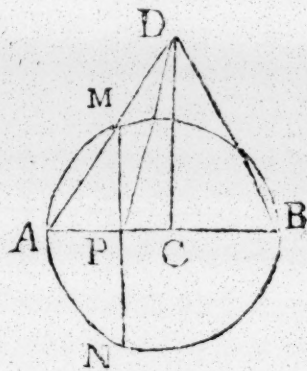
If from the vertex of an equilateral triangle erected upon the diameter of a circle, a right line be drawn to any point in that diameter, and from that point be drawn a right line perpendicular to the diameter meeting the circumference of the circle. The square of that right line, together with the square of that perpendicular, will be equal to the square of the diameter of the circle.

Let AMB be a circle whose diameter is AB , and centre C ; and let ADB be an equilateral triangle described upon the diameter AB . Let P be any point taken in AB . Let the right line PD be drawn, and let PM be perpendicular to the diameter AB , meeting the circumference of the circle in M : I say the two squares of PD , PM will be equal to the square of the diameter AB of the circle.

For join CD .

Then because AC is equal to CB ; AD equal to DB , and CD is common; the angles ACD , BCD [by 4. 1.] will be equal to one another; and so each of them will be a right angle. Therefore the square of AD will [by 47. 1.] be

be equal to the two squares of $A C$, $C D$. And since $A D$ is equal to $A B$, by reason of the equilateral triangle, and [by 4. 2.] four times the square of the semidiameter $A C$ is equal to the square of the diameter $A B$: Therefore four times the square of $A C$, will be equal to the two squares of $A C$, $C D$; and taking away the common square of $A C$ from both, there will remain thrice the square of $A C$ equal to the square of $C D$. Again, because [by 35. 3.] the square of $P M$ is equal to the rectangle under $A P$, $P B$ (for if $P M$ be continued to meet the circumference in N , $M N$ [by 3. 3.] will be bisected in P) and since $A B$ is divided equally in C and unequally in P ; and so [by 5. 2.] the rectangle under $A P$, $P B$, together with the square of $P C$, is equal to the square of $A C$: Therefore the squares of $P C$, $P M$ will be equal to the square of $A C$.



Wherefore since it has been proved that the square of $C D$ is equal to thrice the square of $A C$, if equals be added to equals, the three squares of $C D$, $P C$, $P M$ will be equal to four times the square of $A C$, that is, to the square of $A B$. But because [by 47. 1.] the square of $P D$ is equal to the two squares of $P C$, $C D$, by adding again equals to equals, the squares of $P C$, $P M$, $C D$, $P D$ will be equal to the squares of $A B$, $P C$, $C D$, and taking away the squares of $P C$, $C D$ from both, there will remain the squares of $P D$, $P M$ equal to the square of the diameter $A B$.

Therefore, &c. Which was to be demonstrated.

SCHOLIUM.

There are many other elegant and useful theorems concerning the equalities of the squares and rectangles of right lines drawn in and about a circle, a few of which I shall add at the end of the sixth book, because their demonstrations are shorter and much easier from the proportionality of the sides of equiangular triangles, and the equality of the rectangles under the means and extremes, than by the propositions of the second book.

*I shall here only just mention two theorems; one of Mr. Huygens's, whose demonstration is too long to be subjoin'd. If a circle be described about the centre * of gravity of any right lined figure, and right lines be drawn from all the angles of the figure to any point of the circumference of the circle, the sum of the squares of all those lines will always be of the same magnitude.*

And the following theorem, which was found out by me some years ago, and communicated by me to several people some years ago, viz. The square of the number expressing the area of any trapezium inscribed in a circle, will be equal to the product of the four numerical differences between the number expressing half the sum of the four sides, and each of the numbers expressing the sides. The demonstration, which I have by me, cannot here be conveniently annex'd.

** That point upon which if the figure any how rests, supposing it to be heavy, it would continue without altering its situation.*

E U C L I D's

E L E M E N T S.

B O O K IV.

D E F I N I T I O N S.

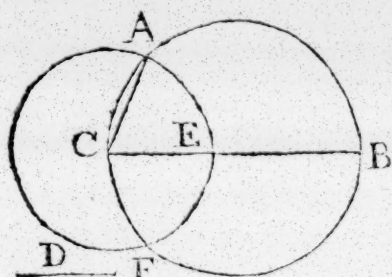
1. **A** Right lined figure is said to be inscribed in a right lined figure, when every angle of the figure inscribed touches every side of the figure in which it is inscribed.
2. A right lined figure is said to circumscribe a right lined figure, when every side of the circumscribed figure touches every angle of the figure, about which it is circumscribed.
3. A right lined figure is said to be inscribed in a circle, when every angle of the inscribed figure touches the circumference of the circle.
4. A right lined figure is said to be circumscribed about a circle, when every side of the circumscribed figure touches the circumference of the circle.
5. A circle is said to be inscribed in a right lined figure, when the circumference of the circle touches every side of the figure in which it is inscribed.
6. A circle is said to be circumscribed about a right lined figure, when the circumference of the circle touches every angle of the figure, about which it is circumscribed.
7. A right line is said to be applied in a circle, when the extremes of that line are in the circumference of the circle.

PROPO-

PROPOSITION I. PROBLEM.

To apply a right line in a given circle equal to a given right line which is not greater than the diameter of the circle^a.

Let the given circle be ABC , and D a given line not greater than the diameter of the circle: it is required to apply a right line in the circle ABC equal to the given right line D .



Draw the diameter BC of the circle ABC ; then if BC be equal to D , the thing required will be done already; for the right line BC is applied in the circle ABC equal to the given right line D . But if not BC is greater than D , and [by

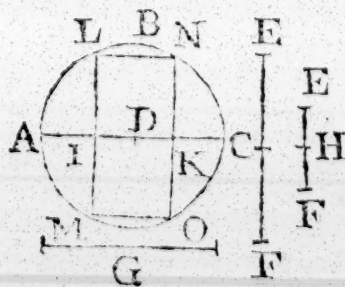
3. 1.] make CE equal to D , and from the centre C with the distance CE [by 3. post.] describe the circle AEF , and [by 1. post.] join CA .

Then because the point C is the centre of the circle AEF , CA will be equal to CE : But D is equal to the right line CE ; and therefore D will be equal to CA .

Wherefore the right line CA is applied in the circle ABC , equal to the given right line D , which is not greater than the diameter of the circle. Which was to be done.

^a If it be required to accommodate or apply a right line less than the diameter of a given circle within the circle equal to a given right line which shall be parallel to another given right line, the thing may be done thus:

Let ABC be the given circle, D the centre, and EF the right line, to which the given right line less than the diameter



of the circle is to be accommodated in the circle equal; and let G be the right line to which the applied right line is to be parallel. Thro' the centre D draw [by 31. 1.] a diameter AC of the circle parallel to the right line G : Then if the right line EF be equal to the diameter, the thing required is done. But if EF be less than the diameter,

diameter, having bisected it in H , cut off DI equal to EH , and DK equal to HF : So that the whole IK be equal to the whole EF ; and thro' I, K [by 11. 1.] draw LM, NO perpendicular to AC ; and join LN : I say LN is applied equal to EF , and parallel to G : For since LM, NO are equally distant from the centre; they will [by 14. 3.] be equal to one another; and because they are divided in half at I, K [by 3. 3.] for they are cut at right angles by the right line AC passing thro' the centre D , their halves LI, KN will be equal. But because [by 28. 1.] LI, NK are also parallel; also LN, IK [by 31. 1.] will be equal and parallel: Wherefore since IK is equal to EF and parallel to G ; LN will be equal to EF , and [by 30. 1.] parallel to G : By the same reason if MO be drawn, it will also be equal to EF , and parallel to G . Which was to be demonstrated.

PROP. II. PROBL.

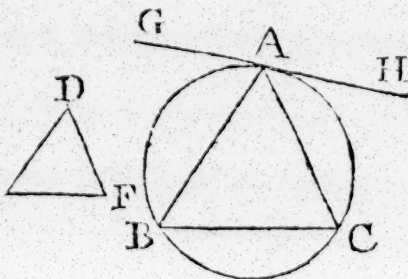
To inscribe a triangle in a given circle, equiangular to a given triangle^b.

Let the given circle be ABC , and the given triangle be DEF ; it is required to inscribe a triangle in the circle ABC equiangular to the triangle DEF .

Draw the right line GAH touching the circle ABC in the point A , and at the right line AH , and at the point A in it [by 23. 1.] make the angle HAC equal to the angle DEF : Also at the right line GA , and at the point A in it make the angle GAB equal the angle FDE , and join BC .

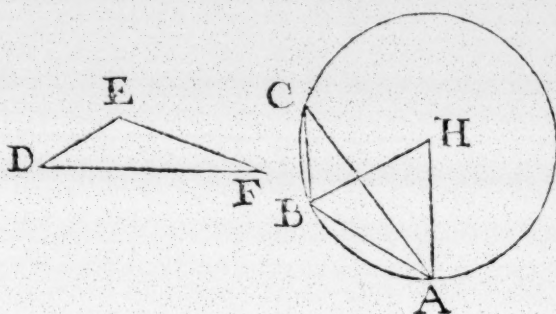
Then because the right line HAG touches the circle ABC , and AC is drawn from the point of contact; the angle HAC [by 32. 3.] will be equal to the angle ABC , which stands in the alternate segment of the circle: But the angle HAC is equal to the angle DEF ; and therefore the angle ABC is equal to the angle DEF . By the same reason, the angle ACB is equal to the angle FDE : Therefore the remaining angle BAC will be [by 32. 1.] equal to the remaining angle EFD ; and so the triangle ABC is equiangular to the triangle DEF , and [by 3. def. 4.] is inscribed in the circle ABC .

Wherefore



Wherefore a triangle is inscribed in a circle, equiangular to a given triangle. Which was to be done.

^b This problem may be otherwise constructed thus: Let ABC be the given circle, whose centre is H , and DEF a given triangle to which a triangle is to be inscribed equiangular to the given triangle DEF : Draw any semidiameter HA , and at the centre H with the semidiameter HA make [by 23. 1.] the angle AHB double to one of the angles EDF of the given triangle. Apply the right line AB . At the point A with the right line AB make the angle BAC equal to the angle DFE of the given triangle. Apply the right line BC . Then will the triangle ABC be inscribed in the given circle ABC equiangular to the given triangle DEF .



For since [by construction] the angle BHA at the centre is double to the angle EDF ; and the angle BCA at the circumference [by 20. 3.] is one half that angle BHA ; therefore the angle BCA will be equal to the angle EDF . But the angle BAC [by construction] is equal to the angle DFE . Therefore [by 32. 1.] the remaining angle ABC will be equal to the remaining angle DEF of the given triangle. Wherefore the triangle ABC is equiangular to the triangle DEF , and is inscribed in the circle ABC . Which was to be done.



Hence an equilateral triangle may be easily inscribed in a given circle; for it is but applying AD equal to the semidiameter AH of the circle; and then again applying BD equal to AH , and joining the right line AB , which will be one side of the equilateral triangle ABC inscribed in the circle. The demonstration is easy, from what has been said above, and from prop. xxvii, xxviii. lib. iii.

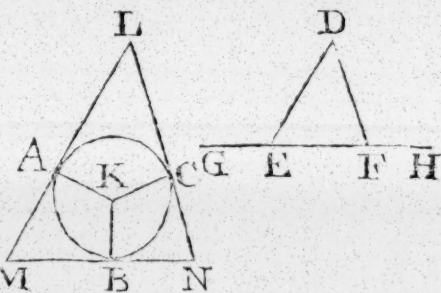
PROP. III. PROBL.

To circumscribe a triangle about a given circle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle;

angle; it is required to circumscribe a triangle about the given circle ABC , equiangular to the given triangle DEF .

Produce EF [by 2. post.] both ways to the points H and G , and [by 1. 3.] find the centre K of the circle ABC , and any how drawing the right line KB , make [by 23. 1.] at the line KB , and at the point K in it, the angle BKA equal to the angle DEG , and the angle BKC equal to the angle DFH ; and draw [by 17. 3.] the right lines LAM , MBN , NCL touching the circle ABC in the points A, B, C .



Then because the right lines LAM , MBN , NCL touch the circle ABC in the points A, B, C , and the right lines KA , KB , KC are drawn from the centre K to the points A, B, C , the angles at the points A, B, C , will [by 18. 3.] be right angles; and because the four angles of the quadrilateral figure $AMBK$, are [by 32. 1.] equal to four right angles (for the quadrilateral figure $AMKB$ may be divided into two triangles) whereof the angles KAM , KBM are each one right angle; the remaining angles AKB , AMB will be equal to two right angles. But [by 13. 1.] the angles DEG , DEF are also equal to two right angles: Wherefore the angles AKB , AMB , are equal to the angles DEG , DEF . But AKB is equal to DEG : Therefore the remaining angle AMB will be equal to the remaining angle DEF . After the same manner we demonstrate, that the angle LMN is equal to the angle DFE ; and so the remaining angle MNL [by 32. 1.] is equal to the remaining angle EDF : Therefore the triangle LMN is equiangular to the triangle DEF , and [by 4. def. 4.] it is circumscribed about the circle ABC .

Therefore a triangle is circumscribed about a given circle equiangular to a given triangle. Which was to be done.

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PROP. IV. PROBL.

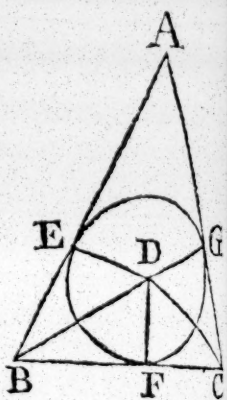
To inscribe a circle in a given triangle.

Let the given triangle be ABC : it is required to inscribe a circle in the triangle ABC .

Divide [by 9. 1.] the angles ABC , BCA each into two equal parts, by the right lines BD , CD , meeting one another in the point D , and from D draw [by 12. 1.] the perpendiculars DE , DF , DG to the sides AB , BC , CA , of the triangle.

Then because the angle ABD is equal to the angle CBD , for the angle ABC is bisected, and the right angle BED is also equal to the right angle BFD , there are two triangles EBD , DBF having two angles of the one equal to two angles of the other, and one side of the one equal to one side of the other, viz. the side BD opposite to one of the equal angles common to them both: Therefore [by 26. 1.] the remaining sides of the one will be equal to the remaining sides of the other; that is, DE will be equal to DF ; also by the same reason the right line DG will be equal to DF : wherefore if a circle be described with the centre D , and either of the distances DE , DF , DG , it will pass thro' the remaining points, and touch the right lines AB , BC , CA , because the angles at E , F , G , are right angles: for if it cuts them the right line drawn from the end of a diameter at right angles to it, will fall within the circle; which [by 16. 3.] is absurd: Therefore the circle described with the centre D and either of the distances DE , DF , DG will not cut the right lines AB , BC , CA ; wherefore it will touch them, and [by 5. def. 4.] be inscribed in the triangle ABC .

Therefore the circle $EF G$ is inscribed in the given triangle ABC . Which was to be done.

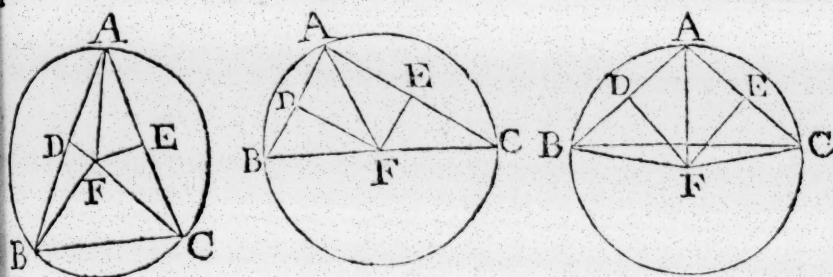


PROP. V. PROBL.

To circumscribe a circle about a given triangle.

Let ABC be a given triangle : it is required to circumscribe a circle about the given triangle ABC .

Divide [by 10. 1.] AB, AC each into two equal parts in the points D, E , and [by 11. 1.] draw DF, EF from the points D, E at right angles to AB, AC ; these will either



meet within the triangle ABC , in the side BC , or without the triangle ABC .

First let them meet within the triangle in the point F , and join BF, FC, FA : Then because AD is equal to DB , and DF is both common and at right angles, the base AF [by 4. 1.] will be equal to the base FB . In like manner we demonstrate that CF is equal to FA , and so BF is equal to FC ; wherefore the three right lines FA, FB, FC are equal to one another: Consequently a circle described from the centre F with either of the distances FA, FB, FC will pass thro' the remaining points; and [by 6. def. 4.] the circle will be circumscribed about the triangle ABC . Let it be circumscribed as ABC .

Secondly, Let DF, EF meet at F in the right line BC , as in the second figure; and join AF . We demonstrate after the same manner that the point F is the centre of a circle circumscribed about the triangle ABC .

Lastly, Let DF, EF meet in the point F without the triangle ABC , as in the third figure; and join AF, FB, FC : Then because AD is equal to DB , and DF is common and at right angles, the base AF [by 4. 1.] will be equal to the base FB . Also after the same manner we demonstrate that CF is equal to FA ; and so BF is equal to FC ; therefore again a circle described with the centre F , and either of the distances FA, FB, FC will pass thro' the remaining points, and will be circumscribed about the triangle

angle ABC . Let it be described as ABC .

Therefore a circle is circumscribed about a given triangle. Which was to be done.

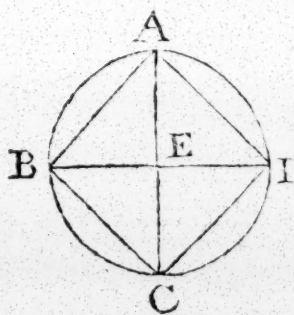
Corollary. Hence it is manifest, that if the centre of the circle falls within the triangle ABC , the angle BAC being in a segment greater than a semicircle, is less than a right angle; if it falls in the right line BC , the angle in the semicircle will be a right angle; and if it falls without the triangle ABC , the angle in the segment less than a semicircle will be greater than a right angle. Wherefore if the given triangle ABC be an acute angled one, DE, EF will meet within the triangle; but if it has a right angle BAC , they will meet in the side BC : If BAC be an obtuse angle, they will meet without the triangle ABC .

PROP. VI. PROBL.

To inscribe a square in a given circle.

Let the given circle be $ABCD$: it is required to inscribe a square in the circle $ABCD$.

Draw [by 11. 1.] two diameters AC, BD at right angles to one another; and join AB, BC, CD, DA .



Then because BE is equal to ED , for E is the centre, and EA is common and at right angles, the base AB [by 4. 1.] will be equal to the base AD ; and by the same reason BC, CD are each equal to BA, AD ; therefore the quadrilateral figure $ABCD$ is equilateral I say it is also right angled.

For because the right line BD is a diameter of the circle $ABCD$; BAD will be a semicircle: and so [by 31. 3.] the angle BAD is a right angle: by the same reason, each of the angles ABC, BCD, CDA will be a right angle. Therefore the quadrilateral figure $ABCD$ is right angled. And it has been proved to be equilateral too: Therefore it is a square, and inscribed in the circle $ABCD$.

Therefore the square $ABCD$ is inscribed in the given circle $ABCD$. Which was to be done.

PROP.

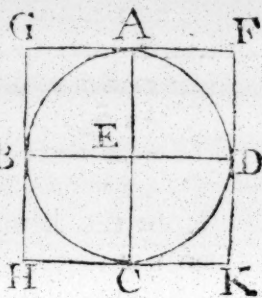
PROP. VII. THEOR.

To circumscribe a square about a given circle.

Let the given circle be $ABCD$. It is required to circumscribe a square about the given circle $ABCD$.

Draw two diameters AC , BD of the circle $ABCD$ at right angles to one another, and [by 17. 1.] thro' the points A , B , C , D , draw the right lines FG , GH , HK , KF , to touch the circle $ABCD$ in the points A , B , C , D .

Then because FG touches the circle $ABCD$, and the right line EA is drawn from the centre E to the point of contact A ; the angles at A [by 18. 3.] will be right angles.



By the same reason, the angles at the points B , C , D are right angles; and because the angle AEB is a right angle, and EBG also a right angle; the line GH [by 28. 1.] will be parallel to AC : By the same reason AC is parallel to FK . In like manner we demonstrate that GF , HK are each parallel to BED . Therefore GK , GC , AK , FB , BK are parallelograms; and so GF is [by 34. 1.] equal to HK , as also GH to FK ; and because AC is equal to BD ; but AC is equal to each of the lines GH , FK , and BD is equal to each of the lines GF , HK : Therefore each of the right lines HG , FK will be equal to each of the right lines GF , HK ; and so the quadrilateral figure $FGHK$ is equilateral. I say it is also rectangular. For because $GBEA$ is a parallelogram, and AEB is a right angle, [by 34. 1.] AGB will be a right angle. After the same manner we demonstrate that the angles at the points H , K , F are right angles; and so the quadrilateral figure $FGHK$ is right angled. But it has been proved to be equilateral; Therefore it is a square, and is circumscribed about the circle $ABCD$.

Therefore a square is circumscribed about a given circle. Which was to be done.

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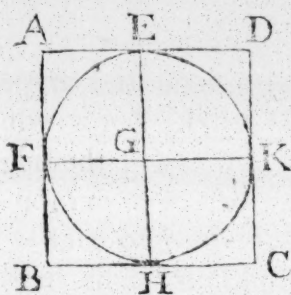
PROP.

PROP. VIII. PROBL.

To inscribe a circle in a given square.

Let the given square be $ABCD$. It is required to inscribe a circle in the given square $ABCD$.

Divide [by 10. 1.] the sides AB , AD into equal parts, in the points F , E ; and thro' E draw [by 31. 1.] EH parallel to AB or CD , and thro' F draw FK parallel to AD or BC .



Then will AK , KB , AH , HD , AG , GC , BC , GD , be each of them parallelograms; and [by 34. 1.] their opposite sides are equal. And because AD is equal to AB , and AE is one half of AD , and AF one half of AB ; AE will be equal to AF : and the sides opposite to them are equal; therefore FG is equal to GE . In like manner we demonstrate that GH , GK are each equal to FG , GE . Therefore the four right lines GE , GF , GH , GK are equal to one another; and so a circle described with the centre G , and either of the distances GE , GF , GH , GK , will also pass thro' the remaining points, and will touch the right lines AB , BC , CD , DA , because the angles at E , F , H , K are right angles: For if the circle cuts the right lines AB , BC , CD , DA , a right line drawn from the end of the diameter of a circle at right angles, will fall within the circle, which [by 16. 3.] is absurd. Therefore a circle described with the centre G , and either of the distances GE , GF , GH , GK will not cut the right lines AB , BC , CD , DA . It will therefore touch them, and [by 5. def. 4.] will be inscribed in the square $ABCD$.

Therefore a circle is inscribed in a given square. Which was to be done.

PROP. IX. PROBL.

To circumscribe a circle about a given square.

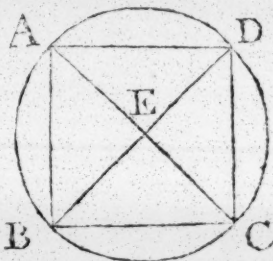
Let the given square be $ABCD$. It is required to circumscribe a circle about the square $ABCD$.

For

For join AC , BD , cutting one another in the point E .

Then because DA is equal to AB , and AC is common, the two sides DA , AC are equal to the two sides BA , AC , and the base DC equal to the base BC ; [by 8. 3.] the angle DAC will be equal to the angle

BAC : Therefore the angle DAB is bisected by the right line AC . After the same manner we demonstrate that each of the angles ABC , BCD , CDA , is bisected by the right lines AC , DB .



Therefore because the angle DAB is equal to the angle ABC , and the angle EAB is one half of the angle DAB , and the angle EBA one half of the angle ABC ; the angle EAB will be equal to the angle EBA : Wherefore [by 6. 1.] the side EA is equal to the side EB . In like manner we demonstrate that the right lines EC , ED are each equal to each of the right lines EA , EB . Therefore the four right lines EA , EB , EC , ED are equal to one another. Wherefore a circle described with the centre E , and either of the distances EA , EB , EC , ED will pass thro' the rest of the points, and will be circumscribed about the square $ABCD$. Let it be described, as $ABCD$.

Therefore a circle is circumscribed about a given square. Which was to be done.

PROP. X. PROBL.

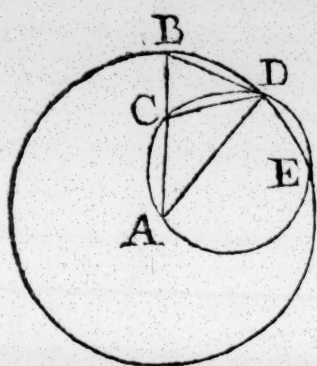
To make an isosceles triangle having one of the angles at the base double to the remaining angle.

Let AB be any given right line, and [by 11. 2.] divide it in the point C , so that the rectangle under AB , BC be equal to the square of CA ; and about the centre A with the distance AB [by 3. post.] describe a circle BDE , and [by 4. 1.] apply the right line BD in the circle BDE , which is not greater than the diameter, equal to CA , and joining DA , DC circumscribe the circle ACD [by 5. 4.] about the triangle ACD .

Then because the rectangle under AB , BC is equal to the square of AC ; and AC is equal to BD ; the

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rectangle



rectangle under AB, BC will be equal to the square of BD ; and because the point B is taken without the circle ACD , and two right lines BCA, BD drawn from the point B fall upon the circle ACD , one of which cuts the circle, and the other meets it, and the rectangle under AB, BC is equal to the square of BD : the right line BD

[by 37. 3.] will touch the circle ACD ; and therefore because BD is a tangent, and DC is drawn from the point of contact D ; the angle BDC [by 32. 3.] will be equal to the angle DAC in the alternate segment of the circle; and since the angle BDC is equal to DAC ; add CDA , which is common to both; then the whole angle BDA is equal to the two angles CDA, DAC . But the outward angle BCD [by 32. 1.] is equal to CDA, DAC ; and therefore BDA is equal to BCD . But the angle BDA [by 5. 1.] is equal to the angle CBD , because the side AD is equal to the side AB : and therefore DBA will be equal to BCD . Wherefore the three angles BDA, DBA, BCD are equal to one another: And because the angle DBC is equal to the angle BCD , the side BD [by 6. 1.] will be equal to the side DC . But BD is made equal to CA ; therefore AC is equal to CD ; and so the angle CDA [by 5. 1.] is equal to the angle DAC : Therefore the angles CDA, DAC taken together are double to the angle DAC : But [by 32. 1.] the angle BCD also is equal to the angles CDA, DAC : Therefore BCD is double to DAC . But BCD is equal to either of the angles BDA, DBA ; and therefore either BDA or DBA is double to DAB .

Therefore there is described an isosceles triangle ADB , having either of the angles at the base BD , double to the remaining angle. Which was to be done.

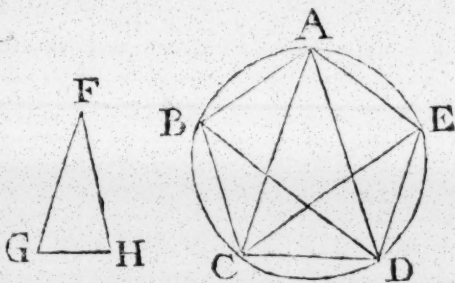
PROP. XI. PROBL.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let the given circle be $ABCDE$. It is required to inscribe an equilateral and equiangular pentagon in the circle $ABCDE$.

Let

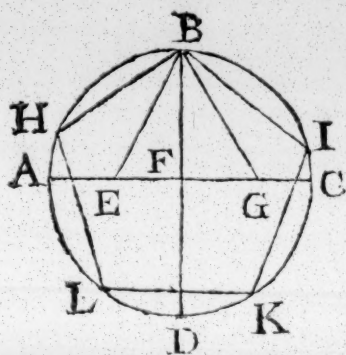
Let [by 10. 4.] FGH be an isosceles triangle having either of the angles at G, H double to the remaining angle at F : and [by 2. 4.] inscribe the triangle ACD in the circle $ABCDE$ equiangular to the triangle FGH ; so that the angle CAD is equal to the angle F , the angles ACD, CDA equal to either of the angles at G, H . Then either of the angles ACD, CDA is double to the angle CAD . Now [by 9. 1.] bisect each of the angles ACD, CDA by the right lines CE, DB , and draw AE, BC, DE, EA .



Then because either of the angles ACD, CDA is double to the angle CAD , and they are bisected by the right lines CE, DB : the five angles DAC, ACE, ECD, CDB, BDA are equal to one another. But [by 26. 3.] equal angles stand upon equal parts of the circumference. Therefore the five parts AB, BC, CD, DE, EA of the circumference are equal to one another. But [by 29. 3.] equal parts of the circumference are joined by equal right lines. Therefore the five right lines AB, BC, CD, DE, EA are equal to one another: Therefore $ABCDE$ is an equilateral pentagon: I say it is also equiangular. For because the part AB of the circumference is equal to the part DE : add the common part BCD , then will the whole $ABCD$, be equal to the whole $EDCB$. But the angle AED stands upon the part $ABCD$ of the circumference, and the angle BAE upon the part $EDCB$. Therefore [by 27. 3.] the angle BAE is equal to the angle AED . By the same reason each of the angles ABC, BCD, CDE is equal to either of the angles BAE, AED . Therefore the pentagon $ABCDE$ is equiangular. And it has been proved to be equilateral too.

Therefore an equilateral and equiangular pentagon is inscribed in a given circle. Which was to be done.

* The following elegant way of describing a regular pentagon in a given circle (to be found in lib. i. of the great construction of Ptolemy) is in reality much shorter than Euclid's way. But as its demonstration depends upon the thirteenth book, Euclid himself, if he had known this method, could



as also IK, KL, LH : And the regular pentagon $BIKLH$ will be inscribed in the circle $ABCD$.

Note also that FG is the side of a regular decagon or figure of ten equal sides and angles inscribed in that circle. The demonstration easily follows from the ninth and tenth propositions of the thirteenth book of Euclid.

PROP. XII. PROBL.

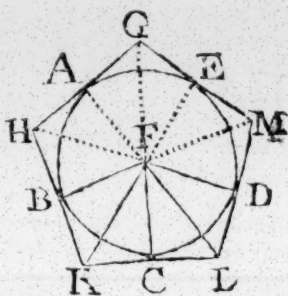
To circumscribe an equilateral and equiangular pentagon about a given circle.

Let the given circle be $ABCDE$. It is required to circumscribe an equilateral and equiangular pentagon about the circle $ABCDE$.

Let the points of the angles of a pentagon inscribed [by 11. 4.] in the circle be supposed to be A, B, C, D, E , so that the parts of the circumference AB, BC, CD, DE, EA be equal: And [by 17. 3.] thro' the points A, B, C, D, E draw the right lines GH, HK, KL, LM, MG to touch the circle: find its centre F : and join FB, FK, FC, FL, FD .

Then because the right line KL touches the circle $ABCDE$ in the point c , and Fc is drawn from the centre F to the point of contact c ; the line Fc [by 18. 3.] will be perpendicular to KL . And so each of the angles at c will be a right angle. By the same reason the angles at the points B, D are right angles. And because the angle FCK is a right angle, the square of FK [by 47. 1.] is equal to the squares of Fc, cK . And by the same reason the square of FK is equal to the squares of FB, BK : Therefore the squares of Fc, cK are equal to the squares of FB, BK ; but the square of Fc is equal to the square of FB : Therefore the remaining square of cK will be equal

equal to the remaining square of BK : and so BK is equal to CK . And because FB is equal to FC , and FK is common, the two sides BF , FK are equal to the two sides CF , FK ; and the base BK is equal to the base CK : Therefore [by 8. 1.] the angle BFK will be equal to the angle KFC , and the angle BKF to the angle FKC : Therefore the angle BFC is double to the angle KFC : And the angle BKC double to the angle KFC . By the same reason the angle CFD is double to the angle CFL , and the angle CLD double to the angle CFL . And because the part BC of the circumference is equal to the part CD of it; the angle BFC [by 27. 3.] will be equal to the angle CFD . But the angle BFC is double to the angle KFC , and the angle DFC double to the angle LFC . Therefore the angle KFC is equal to the angle CFL . Wherefore FKC , FLC are two triangles having two angles of the one equal to two angles of the other, and one side of the one equal to one side of the other, viz. FC which is common to them both; therefore [by 26. 1.] the remaining sides of the one will be equal to the remaining sides of the other, and the remaining angle of the one equal to the remaining angle of the other: Wherefore the right line KC will be equal to the right line CL , and the angle FKC equal to the angle FLC . And because KC is equal to CL ; KL will be double to KC . By the same reason HK will be proved to be double to BK . Again, because BK has been proved to be equal to KC , and KL is double to KC , and HK to BK ; HK will be equal to KL . After the manner we demonstrate that GH , GM , ML are each of them equal to HK , or KL : therefore the pentagon $G HKLM$ is equilateral. I say it is also equiangular. For because the angle FKC is equal to the angle FLC , and it has been proved that HKC is double to the angle FKL ; and DLC is double to the angle KLF : the angle HKL will be equal to the angle KLM . We demonstrate after the same manner that the angles KHG , HGM , GML are each equal to the angles HKL , KLM : Therefore the five angles $G HK$, HKL , KLM , LMG , MGH are equal to one another. Wherefore the pentagon $G HKLM$ is equiangular. But it has also been



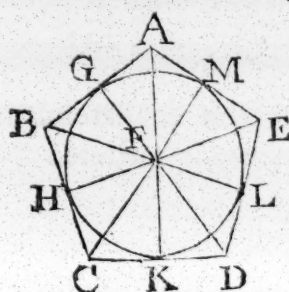
proved to be equilateral, and is circumscribed about the circle $A B C D E$. Which was to be done.

PROP. XIII. PROBL.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let the given equilateral and equiangular pentagon be $A B C D E$. It is required to inscribe a circle in the pentagon $A B C D E$.

Bisect [by 9. 1.] each of the angles $B C D$, $C D E$ by the right lines $C F$, $D F$. And from the point F wherein they meet, draw the right lines $F B$, $F A$, $F E$. And because



$B C$ is equal to $C D$, and $C F$ is common, the two sides $B C$, $C F$ are equal to the two sides $D C$, $C F$, and the angle $B C F$ is equal to the angle $D C F$: Therefore the base $B F$ is [by 4. 1.] equal to the base $D F$, and the triangle $B F C$ is equal to the triangle $D C F$, and the remaining

angles equal to the remaining angles, which are opposite to the equal sides: Therefore the angle $C B F$ will be equal to the angle $C D F$. And because the angle $C D E$ is double to the angle $C D F$, and the angle $C D E$ equal to the angle $A B C$, but the angle $C D F$ equal to the angle $C B F$: The angle $C B A$ will be double to the angle $C B F$; and accordingly the angle $A B F$ equal to the angle $F B C$: Therefore the angle $A B C$ is bisected by the right line $B F$. In like manner we demonstrate that the angles $B A E$, $A E D$ are each of them bisected by the right lines $F A$, $F E$. And so from the point F draw $F G$, $F H$, $F K$, $F L$, $F M$ perpendicular to the right lines $A B$, $B C$, $C D$, $D E$, $E A$. And because the angle $H C F$ is equal to the angle $K C F$, but the right angle $F H C$ is equal to the right angle $F K C$: The two triangles $F H C$, $F K C$, will have two angles of the one equal to two angles of the other, and one side of the one equal to one side of the other, viz. the common side $F C$, which is opposite to one of the equal angles: Therefore [by 26. 1.] their remaining sides will be equal, and the perpendicular $F H$ will be equal to the perpendicular

lar

lar to FK . After the same manner we demonstrate that FL, FM, FG are each of them equal to FH or FK : Therefore the five right lines FG, FH, FK, FL, FM are equal to one another. Wherefore a circle described with the centre F , and either of the distances FG, FH, FK, FL, FM will also pass thro' the remaining points, and the right lines AB, BC, CD, DE, EA will touch it, because the angles at G, H, K, L, M are right angles. For if the circle does not touch, but cut them, a right line drawn from one end of its diameter at right angles to that diameter, will fall within the circle. Which [by 16. 3.] has been proved to be absurd. Therefore a circle described with the centre F , and either of the distances FG, FH, FK, FL, FM , will not cut the right lines AB, BC, CD, DE, EA : it will therefore touch them. Let it be described, as $G H K L M$.

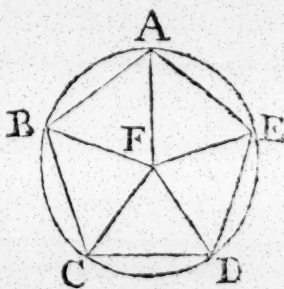
Therefore a circle is inscribed in a given equilateral and equiangular pentagon. Which was to be demonstrated.

PROP. XIV. PROBL.

To circumscribe a circle about a given equilateral and equiangular pentagon.

Let $ABCDE$ be a given equilateral and equiangular pentagon. It is required to circumscribe a circle about the pentagon $ABCDE$.

Bisect [by 9. 1.] each of the angles BCD, CDE by the right lines CF, FD ; and draw the right lines FB, FA, FE from the point F wherein they meet, to the points B, A, E . Then we demonstrate after the same manner as in the last proposition, that the angles CBA, BAE, AED are bisected by the right lines BF, FA, FE . And because the angle BCD is equal to the angle CDE , and the angle BCD is one half of the angle BCD , and CDF one half of the angle CDE ; the angle BCD is equal to the angle FDC : and so [by 6. 1.] the side FC is equal to the side FD . In like manner we demonstrate that FB, FA, FE are each equal to FC or FD : Therefore the five right lines $FA, FB,$



$FC,$

FC, FD, FE are each equal to one another. Wherefore a circle described with the centre F , and either of the distances FA, FB, FC, FD, FE will also pass thro' the remaining points, and be circumscribed about the equilateral and equiangular pentagon $ABCDE$. Let the circle be described, and let it be $ABCDE$.

Therefore a circle is circumscribed about a given equilateral and equiangular pentagon. Which was to be done.

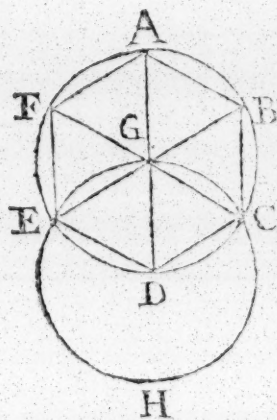
PROP. XV. PROBL.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let the given circle be $ABCDEF$. It is required to inscribe an equilateral and equiangular hexagon in the circle $ABCDEF$.

Draw the diameter AD of the circle $ABCDEF$; and find the centre G of the circle, and about the centre D , with the distance DG describe [by 3. post.] the circle $EGCH$, and joining EG, CG , produce them to the points B, F : Also join AB, BC, CD, DE, EF, FA : I say the hexagon $ABCDEF$ is equilateral and equiangular.

For because the point G is the centre of the circle $ABCDEF$, the line GE will be equal to GD . Again, because D is the centre of the circle $EGCH$; DE will be equal to DG . But GE has been proved to be equal to GD : Therefore GE is equal to DE . Therefore EGD is an equilateral triangle; and so its three angles EGD, GDE, DEG , are equal to one another; because [by 5. 1.] the angles at the base of an isosceles triangle are equal to each other, and [by 32. 1.] the three angles of a triangle are equal to two right angles: Wherefore the angle EGD is one third part of two right angles. After the same manner we demonstrate that the angle DGC is one third part of two right angles. And because [by 13. 1.] the right line CG standing upon the right line EB makes the adjacent angles



EGC,

$\angle EGC, \angle CGB$ equal to two right angles, the remaining angle $\angle CGB$ will be one third part of two right angles: Therefore the angles $\angle EGD, \angle DGC, \angle CGB$ are equal to one another. And so the angles $\angle BGA, \angle AGF, \angle FGE$ at their vertexes [by 15. 1.] are equal to the angles $\angle EGD, \angle DGC, \angle CGB$: Wherefore the six angles $\angle EGD, \angle DGC, \angle CGB, \angle BGA, \angle AGF, \angle FGE$ are equal to one another. But [by 26. 3.] equal angles stand upon equal parts of the circumference: Therefore the six parts of the circumference AB, BC, CD, DE, EF, FA are equal to one another. And [by 29. 3.] equal parts of the circumference are joined by equal right lines. Therefore these six right lines are equal to one another: and so the hexagon $ABCDEF$ is equilateral: I say it is equiangular too. For because the part AF of the circumference is equal to the part ED of it, if the common part $ABCD$ of the circumference be added to both, the whole $FABCD$ is equal to the whole $EDCBA$; but the angled FED stands upon the part $FABCD$ of the circumference, and the angle AFE upon the part $EDCBA$ of it: Therefore the angle AFE [by 27. 3.] is equal to the angle DEF . After the same manner we demonstrate that every one of the remaining angles of the hexagon $ABCDEF$, is equal to either of the angles AFE , or FED . Therefore the hexagon $ABCDEF$ is equiangular: But it has been also proved to be equilateral; and is inscribed in the circle $ABCDEF$.

Therefore an equilateral and equiangular hexagon is inscribed in a given circle. Which was to be done.

Corollary. From hence it is manifest that the side of an hexagon * is equal to the semidiameter of the circle.

And if tangents to the circle be drawn thro' the points A, B, C, D, E, F , an equilateral and equiangular hexagon will be circumscribed about the circle, as is evident from what has been said of the pentagon: And moreover a circle may be inscribed in a given hexagon, or circumscribed about one after the like manner as we have inscribed or circumscribed a pentagon in or about a given circle.

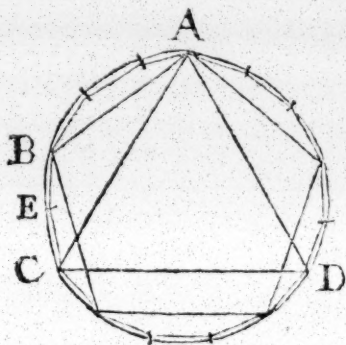
* Inscribed in a circle.

PROP. XVI. PROBL.

*To inscribe an equilateral and equiangular * quindecagon in a given circle^d.*

Let $ABCD$ be the given circle. It is required to inscribe an equilateral and equiangular quindecagon in the circle $ABCD$.

Let AC be the side of an equilateral triangle inscribed in the circle $ABCD$, and AB the side of an equilateral pentagon inscribed in it; then if the circle $ABCD$ be conceived to be divided into fifteen equal parts, the part ABC of the circumference (being one third part of the whole



circumference) will be five of those equal parts. And the part AB of the circumference, being the fifth part of the whole, will be three of those parts; and so the remaining part BC is two of those equal parts [by 30. 3.] Bisect BC in E : Then will BE , EC be each one fifteenth part of the circumference of the circle $ABCD$. If there-

fore the right lines BE , EC be drawn, and right lines equal to them be continually applied in the circle $ABCD$, an equilateral and equiangular quindecagon will be inscribed in the circle. Which was to be done.

In like manner as before for the pentagon, If tangents be drawn thro' the points of division of the circle: an equilateral and equiangular quindecagon will be circumscribed about the given circle: and moreover by proceeding after the like manner as we have done with the pentagon, we may circumscribe or inscribe a circle in a given equilateral or equiangular quindecagon.

* A figure of fifteen sides.

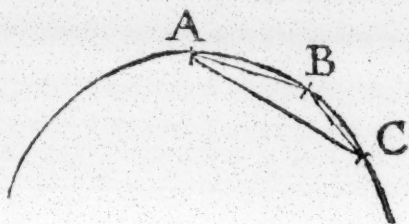
^d It may not be amiss here to observe, first, that every equilateral figure inscribed in a circle, will always be equiangular. But every equilateral figure circumscribing a circle will not necessarily be equiangular, unless it be when the

the number of its angles be odd, or if even, when two angles next to one another are equal, or two not next to one another, when one of them first assumed takes up any other even place, as the fourth, sixth, eighth, &c.

Secondly, that every equiangular figure circumscribed about a circle is also equilateral. But every equiangular figure inscribed within a circle is not necessarily equilateral, unless when the number of its sides is odd; or if even, when two sides next to one another are equal, or two not next to one another equal, when one of them first assumed stands in any even place, as a fourth, sixth, eighth, tenth, &c.

See a full account of this in the scholium of the sixteenth proposition of the fourth book of Clavius's Euclid.

Scarborough says, in his advertisement at the end of this proposition, that a quindecagon admits of no other way of construction but that of Euclid's. But herein I think he is mistaken: For if AC be the side of a regular hexagon inscribed in a circle, also AB the side of a regular decagon inscribed in that same circle. The right line BC applied to the circumference of the circle will be the side of a regular quindecagon inscribed in the circle: because if from the one sixth part of the circumference of the circle be taken away one tenth part, there will remain the one fifteenth part of the circumference.



Addi-

Additions to the Fourth Book.

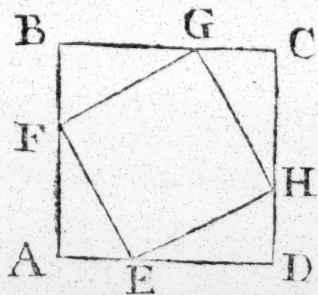
PROP. I. PROBL.

To inscribe a square within a given square, having one of its angles at a given point in one side of the given square.

Let the given square be $ABCD$, and the given point in one of its sides be E . It is required to inscribe a square in the given square $ABCD$, having one of its angles at the given point E .

In the sides AB , BC , CD of the given square, take AF , BG , CH each equal to ED ; and join the points E , F , G , H .

Then because the sides of the square are equal to one another, and AF , BG , CH , DE also are each equal to one another [by construction]. Therefore will AE , BF , GC , DH be equal to one another. And since the four angles EAF , FBG , GCH , EDH of the square are equal to one another, there will be four triangles AFE , BFG , GCH , EDH , having two



sides of the one each equal to two sides of the other, and the angles contained under the equal sides equal too. Wherefore [by 4. I.] the four sides EF , FG , GH , EH will be equal to one another, and so will the four angles AFE , BGF , AEG , BFG , &c. But since the angle FAE is a right angle, and so AEF , AFE [by 32. I.] are equal to one right angle. Therefore AFE , BFG will be equal to one right angle. But the angles AFE , BFG , EFH [by 13. I.] are equal to two right angles. Wherefore the angle GFE will be one half two right angles; that is, will be equal to one right angle. After the same manner we demonstrate that each of the angles FGH , GHE , FHE , will

will be a right angle. Wherefore since the figure $EFGH$ has been proved to be equilateral too. It will be a square; and it is inscribed in the given square $ABCD$.

Therefore, &c. Which was to be demonstrated.

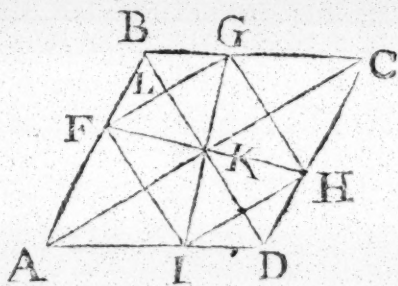
Note, After the same manner may any regular figure be inscribed in another of the same number of equal sides, and at a given point in one of its sides.

PROP. II. PROBL.

To inscribe a square in a given rhombus.

Let there be a given rhombus $ABCD$. It is required to inscribe a square in the given rhombus $ABCD$.

Draw the diagonals AC , BD intersecting one another in the point K . At the diagonal BD , and at the point K in it, make [by 23. 1.] the angle BKG equal to one half of a right angle; and let the side KG of this angle meet the side BC of the rhombus in the point G . In the side BA put BF equal to BG ; in the side AD put AI equal to AF ; and in the side DC put DH equal to DI ; and join the points G , H , I , F . I say the figure $GHIF$ will be a square.



For let FG meet BK in the point L ; and draw GK , FK , IK , HK .

Then because the four sides of the rhombus $ABCD$ are equal to one another: and the diagonals AC , BD are common to the isosceles triangles ABC , ADC ; ABD , BCD ; these triangles [by 4. 1.] will have their respective sides equal, and their angles equal to one another. Therefore the diagonals AC , BD will bisect the opposite angles of the rhombus; and will bisect [by 26. 1.] one another at right angles in the point K . Again, since [by construction] BG is equal to BF , and the angles KBC , KBF have been proved to be equal to one another, each of them being one half the angle ABG , and BL is common; therefore [by 8. 1.] the right lines FL , LG , will be equal to one another, and the angles BLF , BLC equal to one another.

other. Consequently BLF , BLG will be each of them a right angle. And so [by 15. 1.] will the angles FLK , GLK be each a right angle: Again, because FL , LG have been proved to be equal to one another, as also the angles FLK , GLK , and the side KL is common: Therefore [by 4. 1.] the angles LKG , LKF will be equal to one another, and the sides FK , GK equal to one another. But [by construction] the angle LKG is one half a right angle: Therefore the angle LKF will be one half a right angle; and so the angle FKG will be a right angle. After the same manner we demonstrate that IHK will be equal to FG ; IK equal to KH , and IK , FK , GK , KH equal to one another, and the angle IKH a right angle.

Again, because AF , AI are [by construction] equal to GC , CH , and the opposite angles BAD , BCD of the rhombus are [by 34. 1.] equal to one another; the side FI , will be [by 26. 1.] equal to GH , and since FK , KI , KG , KH have been proved to be equal to one another: Therefore the angles FKI , GKH [by 4. 1.] will be equal to one another. Wherefore since [by 13. 1.] all the angles about the same point are equal to four right angles, and it has been proved that the angles FKG , IKH are each right angles: The angles FKI , GKH will also be each a right angle: Wherefore FG or IHK will be equal to FI or GH . And so the figure $IFGH$ will be equilateral. Lastly, because FK , KI are equal to one another, and FKI is a right angle, the angles KFI , FIK will be [by 5. and 32. 1.] each half a right angle. In like manner the angles KFG , FGK will be each one half a right angle; and so will KGH , GKH be each one half a right angle, as well as the angles KHI , KIH : Wherefore the angle IFG will be a right angle. So also will the angles FGH , GHI , HIF be each a right angle. Wherefore the figure $IFGH$ is rectangular. But it has been proved to be equilateral. Therefore it is a square, and is inscribed in a given rhombus.

Therefore, Q.E.D. Which was to be demonstrated.

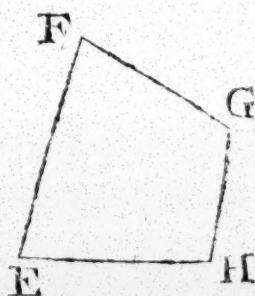
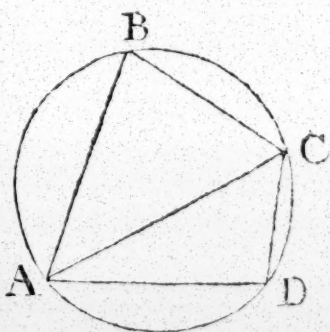
PROP. III. PROBL.

To inscribe a trapezium in a given circle equiangular to a given trapezium, and one of whose sides shall be of a given length. But this length must be less than the diameter of the given circle, and two of the opposite angles of the given trapezium must be equal to two right angles.

Let the given circle be $ABCD$, the given right line IK , less than the diameter of the circle, and the given trapezium $EFGH$, having two of its opposite angles EFG , EHG equal to two right angles; it is required to inscribe a trapezium in the circle $ABCD$ equiangular to the given trapezium $EFGH$ having one of its sides equal to the given right line IK .

From the given circle $ABCD$ cut off [by 34. 3.] the segment ABC containing an angle equal to the angle EFG of the given trapezium $EFGH$, draw the right line AD [by 1. 4.] e-

$I \text{ --- } K$



qual to the angle FEH of the given trapezium $EFGH$; and let the side AB of this angle cut the circumference of the given circle in the point B . Join the points B, C, D . And the trapezium $ABCD$ will be that required.

For because all the angles in the same segment of a circle [by 21. 3.] are equal to one another, the angle ABC will be equal to the angle EFG , since [by construction] the segment ABC of the circle contains an angle equal to the angle EFG . And because [by 22. 3.] the angles ABC , ADC are equal to two right angles, and the angles EFG , EHG of the given trapezium are also [by supposition] equal to two right angles. Therefore will the angle ADC ,
be

be equal to the angle $E H G$ of the given trapezium $E F G H$. But the angle $B A D$ [by construction] is equal to the angle $H E F$ of the trapezium. Wherefore the three angles $A B C$, $A D C$, $B A D$ of the trapezium $A B C D$ will be respectively equal to the three angles $E F G$, $E H G$, $F E H$ of the given trapezium $E F G H$. And so because all the angles of any trapezium are [by 3. addit. to lib. 1.] equal to four right angles, the fourth angle $B C D$ of the trapezium $A B C D$ will be equal to the fourth angle $F G H$ of the given trapezium $E F G H$. Therefore the trapezium $A B C D$ will be equiangular to the given trapezium, having one side $A D$ equal to the given right line $I K$; and it is inscribed in the given circle $A B C D$.

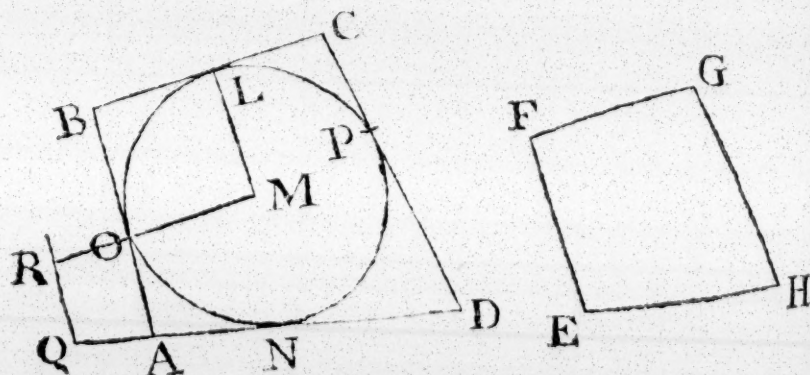
Therefore, &c. Which was to be done.

PROP. IV. PROBL.

To circumscribe a trapezium about a given circle, equiangular to a given trapezium.

Let $N O L P$ be the given circle, whose centre is M , and let $E F G H$ be a given trapezium. It is required to circumscribe a trapezium about the given circle $N O L P$ equiangular to the given trapezium $E F G H$.

Draw [by 17. 3.] from an assumed point Q without the circle the right line $Q N D$ touching the circle in the point N . At the right line $Q D$, and at the point Q in it, make [by 23. 1.] the angle $D Q R$ equal to the angle E of the given trapezium; and from the centre M of the circle draw [by 12. 1.] the perpendicular $M R$ to $Q R$, the side of the angle meeting the circumference of the circle in the point O ; thro' O draw [by 31. 1.] $A O B$ parallel to $Q R$. Then [by 16. 3.] the right line $A B$ will touch the circle



in o ; and because AO, QR [by construction] are parallel: Therefore [by 29. 3.] the angle OAD will be equal to the angle RQD . But the angle RQD [by construction] is equal to the angle E of the trapezium. Therefore the angle oAN will be equal to the angle E of the trapezium. Again, by a like operation, draw the right line DP to touch the circle in P , making an angle QDC with the right line QD equal to the angle H , of the given trapezium. And by a further like operation draw BC to touch the given circle in L , making the angle ABC with AB equal to the angle F of the given trapezium, and let BC meet DC in the point C . I say $ABCD$ will be the trapezium required.

For it has been already proved that the angle BAD of the trapezium $ABCD$ is equal to the angle E of the given trapezium $EFGH$. So also does it appear [by construction] and from what has been already proved, that the angles ABC, ADC of the trapezium $ABCD$ are each equal to the angles F, H of the given trapezium: Therefore the remaining angle BCD of the trapezium $ABCD$ will be [by 3. addit. to book I.] equal to the angle G of the trapezium $EFGH$. Therefore the trapeziums $ABCD, EFGH$ are equiangular. But [by construction] the trapezium $ABCD$ touches the given circle $OLPN$ in the points O, L, P, N .

Therefore, &c. Which was to be demonstrated.

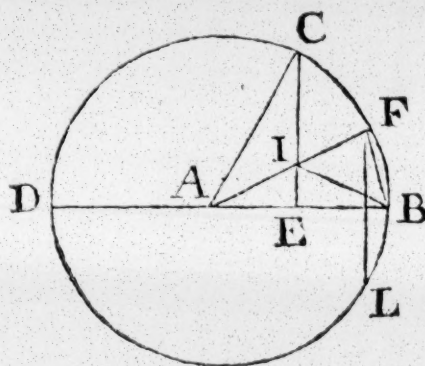
PROP. V. PROBL.

To inscribe a regular heptagon, or figure having seven equal sides, in a given circle, partly by trial.

Let DCB be a given circle, whose centre is A . It is required to inscribe a regular heptagon in the given circle DCB , partly by trial.

Draw the diameter DB , and assume [by 15. 4.] the arch BC of a regular hexagon. From c draw [by 12. 1.] the right line CE perpendicular to the semidiameter AB of the circle, and draw another semidiameter AF [by trial, or turning a ruler about the centre A , &c.] so cutting the circumference of the circle in the point F , and the

perpendicular CE in I , that drawing the right line BF , the same shall be equal to IF . This done, make the



arch BL equal to the arch BF , and draw the right line FL . I say FL will be the side of a regular heptagon inscribed in the given circle DCB . One side FL being thus found, the others will be obtained by carrying that side six times about the circumference of the given circle.

Join BI .

Then since the sides AB , AF are equal to one another, the angles AFB , ABF [by 5. 1.] will be equal to one another. And since [by cor. 15. 4.] the side of the regular hexagon inscribed in the circle is equal to the semidiameter AC ; and because CE is perpendicular to AB , [by constr.] therefore [by 26. 1.] AE will be equal to EB ; AI equal to IB , and the angle IAE equal to the angle IBE . But since IF , FB [by constr.] are equal to one another, the angles FIB , FBI [by 5. 1.] will be equal to one another. But [by 32. 1.] the outward angle FIB of the triangle AIB , to which FBI is equal, is equal to both the inward angles IAB , IBA : To the angle FBI add the angle IBA , that is, FAB . Then will the angle ABF , or AFB equal to it, be thrice the angle FAB . Again, because [by 20. 3.] an angle at the centre of a circle is double to an angle at the circumference, the angle DAF will be double to the angle ABF . And since the angle ABF is three times the angle FAB . Therefore the angle DAF will be six times the angle FAB ; and so the angle FAB will be one seventh part of two right angles. Wherefore [by 26. 3.] the arch FB will be one seventh part of the arch of the semicircle BCD . And accordingly [by construction] the arch FL will be one seventh part of the whole circumference.

Therefore, &c. Which was to be demonstrated.

SCHOLIUM.

This problem is not to be resolved geometrically; that is, by a right line and a circle. Its solution requires a curve line

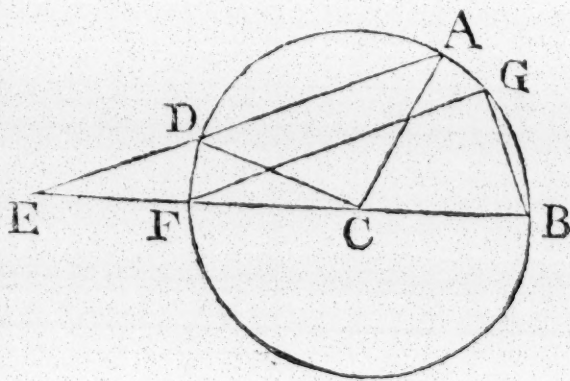
line of an higher order, as a conic section, or the conchoid. It arising, as the algebrists call it, to a cubic equation. For if the semidiameter of the given circle be called r , and the side of the given regular heptagon inscribed in it be called x , one of the roots of this cubic equation $x^3 + 4rxx + 3rrx - r^3 = 0$, will give the value of the side of the regular heptagon inscribed in the circle.

PROP. VI. PROBL.

To inscribe a regular nonagon, or figure of nine equal sides, in a given circle, partly by trial.

Let $F D A B$ be a given circle, whose diameter is $F B$, and centre C . It is required to inscribe a regular nonagon in the given circle $F D A B$.

Draw the diameter $B F$ of the circle, which continue out to E . Take [by 15. 4] the arch $A B$ equal to the sixth part of the whole circumference of the circle. From A draw the right line $A D E$, so cutting the circumference of the circle in the point D , and the continuation of the diameter $B F$ in the point E , that the part $E D$ of the right line $E A$ be equal to the semidiameter $F C$ or $C B$ of the circle. But this is to be done by an easy and expeditious trial. When this is done from the point F , draw [by 31. 1.] the right line $F G$ parallel to the right line $E A$ cutting the circumference of the circle in the point G . Join



$B G$. I say $B G$ will be one side of a regular nonagon, or figure having nine equal sides, inscribed in the given circle $F D A B$; which apply'd nine times round the circumference of the given circle will be that required.

For join the semidiameters $C A$, $C D$.

Then since [by construction] $E D$ is equal to $D C$; the angle $D E F$ [by 5. 1.] will be equal to the angle $D F C$.

O 3

But

But [by 32. 1.] because the angle CDA is equal to DEC , ECD : Therefore will the angle CDA be twice the angle CED . By the same reason, because CD is equal to CA , &c. the angle ACB will be thrice the angle DCE , and so since [by construction] FG is parallel to EA , the angle ACB [by 29. 1.] will be thrice the angle CFB . Therefore [by 26. 3.] the arch AB will be thrice the arch FD ; that is, the arch AB will be thrice the arch AG , for the arch AG is equal to FD , since AD, FG are parallel. Therefore the arch BG will be two equal parts of three of the arch AB , or four equal parts of six of the arch AB ; that is, since AB [by construction] is one sixth part of the circumference of the circle, the arch BG will be four equal parts of thirty six, or two equal parts of eighteen, or one part of nine, of the whole circumference of the circle.

Therefore, &c. Which was to be demonstrated.

Note, This problem, no more than the last, can be resolved by a right line and a circle.

PROP. VII. PROBL.

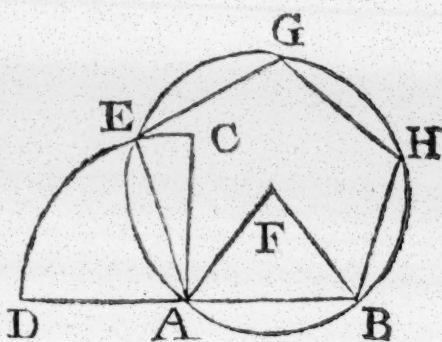
To describe any regular polygon upon a given right line, admitting the division of a given arch of a circle into any number of equal parts.

Let there be a given right line AB . It is required to describe a regular polygon upon the given right line AB .

First, Let the number of sides of the polygon be odd. Continue out the given right line AB to D , so that AD be equal to AB . About the centre A with the distance AD , describe part of a circle DEC . From A [by 11. 1.] draw AC perpendicular to AD , meeting the circumference of that circle in the point C . Divide the quarter DC of the circumference into the same number of equal parts as the polygon to be described has sides. Then if the polygon has five sides, let CE be one of those parts. If it has seven sides, let CE be three of those parts; if the polygon has nine sides, let CE be five of those parts; if it has eleven, let it be seven of those parts; and so on increasing by two, &c. But if the number of sides of the polygon to be described be even, the arch CE , if the poly-

gon has six sides, must be two of the six equal parts the quadrant DC is divided into, or one part of three that it is divided into. If the polygon has eight sides, the arch CE must be four of eight equal parts; that is, one half the quadrant DC . If the polygon has ten sides, the arch CE must be six of ten equal parts, or three of five that the quadrant DC is divided into. If the polygon has twelve sides, the arch CE must be nine parts of twelve, or three parts of four that the quadrant DC is divided into, and so on, &c.

This done, draw the semidiameter AE of the quadrant DC . Bisect [by 9. I.] the angle EAB by the right line AF , and having made the angle ABH [by 23. I.] equal to the angle EAB , bisect the angle ABH by the right line BF meeting the right line AF in the point F , and describe a circle AGB about the centre F with either of the distances AF , FB ; and applying a right line equal to AB or AE from B to H , and from H to G , and from G to E , &c. round the circumference of this circle. I say there will be described upon the given right line AB the regular polygon required.



For because all the angles of a regular polygon are equal to one another; and since the sum of all the angles of any right lined figure whatsoever, is [by prop. 3. of addit. to lib. I.] equal to twice as many right angles as the figure has sides, abating four; therefore all the angles of a pentagon will be equal to six right angles. Consequently one of its angles will be one right, and one fifth part of one right angle. In like manner one of the angles of a regular heptagon will be equal to one right angle, and three parts of seven of one right angle; one of the angles of a nonagon will be one right angle, and five equal parts of nine of one right angle; and one of the angles of a regular polygon having eleven equal sides, will be equal to one right angle and seven equal parts of eleven of one right angle; and so of others. In like manner it is demonstrated that one angle of a regular hexagon will be equal to one right angle

angle, and one third part of one right angle; one angle of a regular octagon equal to one right angle and half a right angle; one angle of a regular decagon equal to one right angle and three fifth parts of one right angle; and one angle of a regular dodecagon equal to one right angle and three fourth parts of one right angle, &c.

Therefore [by construction] the angle EAB will be one angle of the required regular polygon, and ABH another angle of the polygon. And since the semidiameter of a circle drawn to any one angle of a regular polygon inscribed in a circle bisects that angle, therefore the point F will be the centre of a circle, in which the regular polygon required is described, and AF or FB will be a semidiameter, &c.

Therefore, &c. Which was to be done.

PROP. VIII. THEOR.

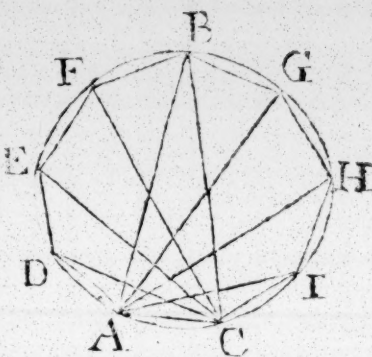
If the base of an isosceles triangle stands upon the arch of a circle, and the vertical angle be in the circumference of that circle, and one angle at the base of that triangle be some number of times the angle at the vertex, and each of the angles at the base be divided into angles each equal to that at the vertex by right lines meeting the circumference of the circle. And if the extremities of these right lines be joined, there will be constituted a regular polygon in the circle consisting of as many sides as there are units in twice the number of divisions of one of the angles at the base of the isosceles triangle, together with a unite added.

Let ABC be an isosceles triangle, whose base AC stands upon the circumference of the circle $AEBHC$, and vertical angle B is also in that circumference; and let one of the angles BAC at the base be four times the angle ABC at the vertex. Let the angles BAC , BCA be divided into four angles each equal to the vertical angle ABC , by the right lines AG , AH , AI ; CF , CE , CD meeting the circumference of the circle in the points G , H , I ; F , E , D .

join

Join the right lines $BG, GH, HI, IC; BF, FE, ED, DA$. I say the polygon $ADEFBGHCIA$ will have nine equal sides.

For because the four angles BAG, GAH, HAI, IAC [by supposition] are each equal to the angle ABC at the circumference, and also the four angles FCB, ECF, ECD, DCA at the periphery each equal to the same angle ABC . Therefore [by 26. and 29. 3.] the nine right lines $BG, GH, HI, IC, AC, AD, DE, EF, FB$ will be equal to one another. And since these are equal to one another, the angles $DAC, ACI, D, E, F, FBG, G, H, I$ will be also equal to one another. Wherefore the polygon $ADEFBGHCIA$ will be a regular nonagon. The demonstration is the same whatever be the number of times that one of the angles at the base of the isosceles triangle ABC contains the angle ABC at the vertex.



Therefore, &c. Which was to be demonstrated.

SCHOLIUM.

From this proposition it is evident, that a regular polygon of any number of sides, might be inscribed in a given circle, if an isosceles triangle could be made such that one angle at the base should contain the angle at the vertex any given number of times. But this is not to be done generally by a right line and a circle; it must be effected in some cases by a conic section, and in others by geometrical curves of an higher order. The order of the geometrical curve necessary to the business, being higher according as the number of times the angle at the base is greater than that at the vertex. But as the construction by these curves is useless, by reason of the difficulty of the solution, the cycloid or spiral of Archimedes, by which the problem can be easily resolved, are much preferable.

PROP. IX. THEOR.

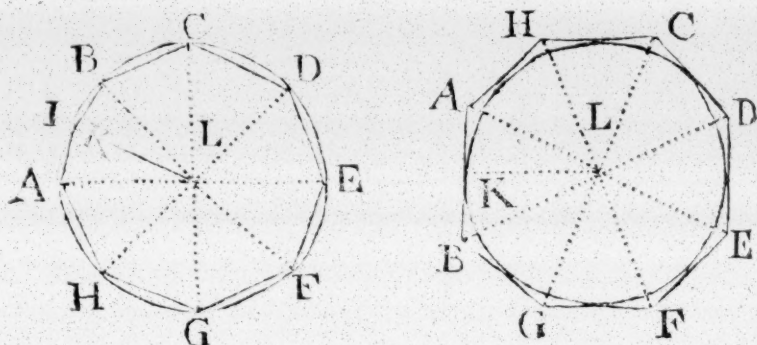
Any regular polygon inscribed in a circle, or circumscribed about a circle, will approach nearer to the circle, according as the number of the sides of the polygon is greater.

Let

Let $ABCDEFGH$ be a regular octagon inscribed in a circle, whose centre is L .

Bisect [by 10. 1.] the side AB in the point K , and draw the semidiameter LI , and join AI , IB .

Then because KA is equal to KB , the angles AKI , BKI [by 3. 3.] are right angles, and the side IK is common; the right lines AI , IB [by 26. 1.] will be equal to one another; and so [by 28. 3.] the arches AI , IB will be equal to one another. Now the circle will exceed the



octagon $ABCDEFGH$ by eight equal segments AIB , BC , CD , DE , EF , GF , HG , AH , and the trilineal space AIK contained under AK , IK , and the arch AI will be one half the segment AIB of the circle, which is evidently greater than either of the equal segments AI , IB by the right angled triangles AIK , or IBK . Wherefore sixteen of the segments AI , whereby the circle differs from an inscribed regular polygon of sixteen equal sides will be less than eight of the segments AB , whereby the circle differs from the regular polygon $ABCDEFGH$ of eight sides. Consequently a regular polygon of sixteen sides is nearer to the circle in which it is inscribed, than that of eight sides. In like manner a regular polygon of thirty two sides inscribed in the circle, will differ less from the circle than one of sixteen sides; and one of sixty four sides will still be nearer to the circle in which it is inscribed, than one of thirty two sides; and so on *ad infinitum*. The demonstration is the same, whatever be the number of the sides of the assumed polygon $ABCDEFGH$ inscribed in the circle. So likewise when a regular polygon circumscribes the circle, the demonstration of the theorem is much the same as when the polygon is inscribed in the circle.

Corollary. Hence when the number of the sides of a regular polygon inscribed in a circle, or circumscribed

about

about a circle are exceeding great, the difference between the circle, and either of those polygons, will be exceeding small, that is all three of them; the circumscribed polygon, the circle and the inscribed polygon will be so nearly equal to one another, as to differ only by a space less than any assignable one; the same may be said of the circumferences or ambits of either of these polygons, and that of the circle; and hence it is, that some have called a circle a polygon of an infinite number of sides.

SCHOLIUM.

Altho' in the proposition we have only taken a regular polygon, yet the thing is true of any polygons soever circumscribing or inscribing a circle. But the demonstration is not so easy.

PROP. X. THEOR.

First, Any regular polygon inscribed in a circle will be equal to a triangle whose base is the circumference of the whole polygon, and perpendicular the right line drawn from the centre of the circle to the middle point of one of the sides of the polygon. And secondly, Any regular polygon circumscribing a circle will be equal to a triangle whose base is the circumference of the whole polygon, and perpendicular the semidiameter of the circle. Thirdly, A circle is equal to a triangle whose base is the circumference of the circle, and perpendicular the semidiameter of the circle.

I. For [see the figure of the last proposition] let $ABCDEFGH$ be a regular octagon, inscribed in or circumscribed about the circle. I say the polygon $ABCDEFGH$ when inscribed in the circle, will be equal to a triangle whose base is the sum of the sides $AB, BC, CD, DE, EF, FG, GH, HA$, and perpendicular the right line LK drawn from the centre L to the middle point K of one side AB of the polygon. And when it is circumscribed, its space will be equal to a triangle whose base will be the sum of the sides of the polygon, and perpendicular equal to the semidiameter IL of the circle.

For

For drawing the several semidiameters $AL, BL, CL, \&c.$ these will divide the polygon into as many equal isosceles triangles as the figure has sides, whose common perpendicular altitude will be the right line LK . Wherefore [by I. 2.] a triangle whose base is the sum of the sides $AB, BC, CD, \&c.$ and perpendicular altitude LK will be equal to the inscribed or circumscribed polygon.

Again, Because [by cor. of the last prop.] a regular polygon of an exceeding great number of sides circumscribing or inscribing a circle differs from the circle itself, by an exceeding small magnitude, *viz.* less than any assignable one, as well as the circumference of such a polygon from that of the circle: Therefore [by what has been already demonstrated] a circle will be equal to a triangle whose base is the circumference of the circle, and perpendicular altitude the semidiameter thereof.

Corollary. Hence a sector of a circle is equal to a triangle, whose base is the arch of the sector, and perpendicular the semidiameter of the circle.

This is the famous proposition of *Archimedes*, in his book of the *Circle*, which he has demonstrated by the method of exhaustions, as it is called.

E U C L I D ' s E L E M E N T S. B O O K V.

D E F I N I T I O N S.

1. **A** Part is a magnitude of a magnitude, a less of a greater, when the less measures the greater.
2. A multiple is a greater [magnitude] of a less, when the greater is measured by the less ^a.
3. Ratio is a certain mutual relation of two magnitudes to one another of the same kind, according to quantity ^b.

^a It might perhaps be better to call *that magnitude any number of times greater than another a multiple*, and *that magnitude any number of times less than another a submultiple*, and not a part, as Euclid has called it; because part generally signifies any magnitude less than a whole; and the word *to measure*, seems as much to want defining as *part*. Some call this an aliquot part.

^b *Ratio*, according to the etymology of the word, signifies a judgment, account, or estimation of things known, from a comparison of them.

In every ratio is usually considered how much the antecedent contains of the consequent, or the consequent of the antecedent. For let the antecedent be either equal to, greater, or less than the consequent, it is the quantum of the consequent contained in the antecedent, which is generally taken for the ratio of the antecedent to the consequent; and as the antecedent contains more or less of its consequent, so it is more or less valued in a respect to that consequent.

There is no obtaining an adequate notion of a ratio from Euclid's definition of it; because there are other comparisons of two magnitudes of the same kind, according to quantity, that are not properly ratio, as the excess whereby one magni-

4. Magni-

4. Magnitudes are said to have a ratio to one another, which being multiplied can exceed each other^c.

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth: When the equimultiples of the first and third compared with the

tude exceeds another, or the defect whereby one magnitude is less than another, is a certain sort of mutual comparison of them, *viz.* as to excess or defect, but this comparison is not their ratio.

As some take the numerical exponent or measure of a ratio to be the quotient of the division of the number expressing the measure of the antecedent term of that ratio by the number expressing the consequent term: As if the magnitude A be the antecedent, and the magnitude B the consequent, and A be three unites, and B be one unite of the same kind; then will the numerical exponent or measure of the ratio of A to B be 3. And if B be the antecedent, and A the consequent, the numerical exponent of the ratio of B to A will be $\frac{1}{3}$.

So there are others who call the logarithms the numerical exponents or measures of ratios, the very meaning of the word implying it, *viz.* the ratio of numbers. And accordingly these take the measures of ratios to be either the abscissies of the logarithmic curve, or the sectors of equilateral hyperbolas expressed in numbers. See Mr. Coates's *Harmonia mensurarum*.

^c Magnitudes of different kinds, such as lines and superficies, angles of contact and right lined angles, superficies and solids, lines and solids, cannot have any ratio to one another; because a line never so often multiplied will never exceed, or ever become a superficies; nor a superficies exceed or become a solid; nor any number of angles of contact become equal to, much less exceed, any the least right lined angle.

Euclid has not expressly told us what the quantity of a ratio is, yet from the fifth definition of the sixth book, one would think he meant by it the quotient of the division of the antecedent term of the ratio by the consequent term. But whether it be taken as this quotient, or whether it be taken as the logarithm of that quotient, there can no error arise from either of these suppositions when they are rightly understood. There has indeed been a good deal of dispute amongst mathematicians about this: But it has chiefly arose from both parties not considering either of those suppositions might be taken at pleasure, and because some having taken it to be the one, and some the other, they have both reasoned accordingly; and so disagreed in their conclusions, as they certainly must, because of their

equi-

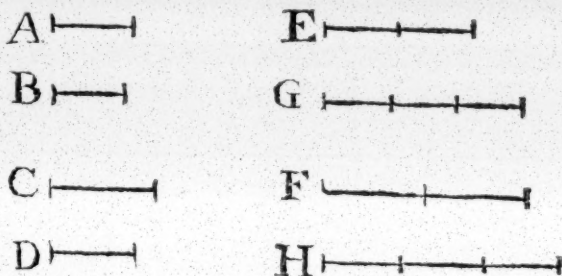
equimultiples of the second and fourth, according to any multiplication whatsoever, are either together deficient or together equal, or together exceeding each other^d.

6. Magnitudes which are in, or have the same ratio, are called proportionals.

different premises. I readily own, that the supposition of the measure of a ratio being a quotient is most simple and easy. But it is more apt to mislead, and perhaps is not quite so accurate as the supposition of its being the logarithm of that quotient, which is a more difficult and remote consideration.

^d Let there be four magnitudes A, B, C, D, where the first A is compared to the second B, and the third C to the fourth D. Let E be any multiple whatsoever of the first A, and F the same multiple of the third C, as let E be the double, triple, quadruple, &c. of A, and F the double, triple, quadruple, &c. of C. And

again let G be any multiple of the second magnitude B, either the same as before E was of A, or F of C, or any other whatsoever; and let H be the same multiple of the fourth



magnitude D, as G is of B.

Then whensoever it is demonstrated that according to any multiplication whatsoever, the equimultiples E, F of the first magnitude A and the third C, compared to the equimultiples G, H of the second B and fourth D, each to each, E to G, and F to H. I say when these equimultiples are proved to be together, less, or equal, or greater, E than G, and F than H. Then these four magnitudes A, B, C, D are said to be in the same proportion A to B, as C to D. And so when in any one particular instance the contrary shall be demonstrated, that the equimultiple of either A exceeds the equimultiple of B, and the other exceeds not, but is either equal or less; Then the proposed magnitudes A, B, C, D are not in the same proportion, the first to the second as the third to the fourth, because the agreement of the equimultiples in a joint defect, equality, or excess ought to hold in any multiplication whatsoever.

Many have disliked this definition of Euclid, and thought it foreign, difficult, and obscure. But this was for want of a thorough understanding thereof. This definition is plainly and clearly deduced from the fourth and fourteenth propositions of the fifth book. For since it is demonstrated in the

7. But

7. But when amongst the equimultiples [of four magnitudes] the multiple of the first [magnitude] shall exceed that of the second, but the multiple of the third shall not exceed that of the fourth; then the first magnitude is said

fourth proposition that any equimultiples of the first and third of four proportional magnitudes, are proportional to any equimultiples of the second and fourth, and because it is shewn in the fourteenth proposition, if four magnitudes be proportional, and the first be greater than the third; the second shall be greater than the fourth; and if equal, equal; if less, less. Therefore when four magnitudes are proportional, if any equimultiple of the first be greater than that of the third, any other equimultiple of the second will be greater than that of the fourth. If equal, equal; if less, less. Wherefore, on the contrary, when there are four proposed magnitudes, and any equimultiples of the first and third be taken, as also any other equimultiples of the second and fourth; and the equimultiple of the first be always greater than that of the third; and that of the second at the same time always greater than that of the fourth; if that of the first be equal to that of the third, that of the second always equal to that of the fourth; or if that of the first be less than that of the third, that of the second always less than that of the fourth. Then it necessarily follows that those four proposed magnitudes will be proportional; so that from hence it most clearly appears how Euclid obtained this definition, and that it is a very simple, natural, and easy sign of proportionality, derived from the before mentioned two propositions. However otherwise it may at first appear to those who will not be at the pains to consider it. It is true it is not so simple and plain as the definition of numbers, or that which might be given of commensurable magnitudes. Nor does it at all agree with the common notion that the generality of mankind conceive of proportionals. Yet in use and practice it is most plain and easy. Euclid could not have given any other so elegant and general a definition that would take in incommensurable magnitudes, as well as numbers and commensurable ones; and therefore he did aright to give this rather than a worse. See Dr. Barrow's full and learned defence of this definition of *Euclid*, in his 21st and 22d Mathematical Lectures; at the end of the latter whereof the Dr. concludes in these words: *There is nothing extant in the whole work of the Elements of Euclid more subtly invented, more solidly established, or more accurately handled, than the doctrine of proportionality.*

Some have given other definitions of proportional magnitudes, as Borellus, in his *Euclides restitutus*; Taquet, in his *Eu-*

to have a greater ratio to the second, than the third has to the fourth^e.

8. Proportion is a similitude of ratios.

clid; and Malcolm, in his *Arithmetic*. But, as I have already said, Euclid's is the best of them all.

I have sometimes thought that the doctrine of proportionality, in all quantities whatsoever, might be easily derived from the following positions, and Euclid's seventh book.

1. *That those quantities that differ from one another by magnitudes less than any assignable magnitudes, may be taken for equals, or represent one another.*

2. *That any four proposed magnitudes may either be accurately expressed in numbers, or else four magnitudes that differ from them by magnitudes less than any assignable ones may; and therefore*

3. *Any four magnitudes may be taken for proportionals, the first to the second, and the third to the fourth, when the product of the multiplication of the numbers representing or measuring the first and fourth is equal to the product of the multiplication of the numbers representing or measuring the second and third magnitudes; and accordingly*

4. *Whatever properties of proportionality are deduced and demonstrated from the definition of proportional numbers, or from Euclid in his seventh book, will hold good of the proportionality of any magnitudes soever which they represent or measure.*

^e Euclid here declares what condition four magnitudes ought to have when the ratio of the first to the second is greater than that of the third to the fourth, saying that taking equimultiples of the first and third, and of the second and fourth. If it shall be at any time found (altho' not always) that the multiple of the first is greater than the multiple of the second; but the multiple of the third not greater than that of the fourth, but less than it, or equal to it. Then the ratio of the first magnitude to the second is said to be greater than the ratio of the third to the fourth, as is manifest by the annex'd example; wherein the magnitudes E, F are taken thrice the first magnitude A, and the third C, and the magnitudes G, H four times the second B, and the fourth D. And because E the multiple of the first magnitude is greater than G, the multiple of the second; but F the multiple of the third is not greater than the multiple of the fourth, but less; the ratio of the first magnitude A to the second B, is said to be greater than the ratio of the third C to the fourth D.



F

9. There

9. There must be three terms at least to constitute proportion^f.

10. If three magnitudes are proportionals, the ratio of the first to the third is said to be the duplicate of the ratio of the first to the second^g.

11. If four magnitudes be [continual] proportionals, the ratio of the first to the fourth is said to be triplicate of the ratio of the first to the second, and so forwards always more by one, as long as the proportion shall be continued.

12. Magnitudes are homologous, when the antecedents are to the antecedents, as the consequents to the consequents^h.

13. Alternate ratio is the assumption of the antecedent to the antecedent, and of the consequent to the consequentⁱ.

^f Both this definition and the last appear to be superfluous, and of no manner of use; and therefore I rather take them to be remarks of some learner put into the text, than genuine definitions of Euclid himself.

^g As subduplicate and subtriplicate ratio are not mentioned by Euclid, and as these are very useful, I thought it might not be amiss to define them here. Accordingly if three magnitudes be proportional, the ratio of the first to the second, or that of the second to the third, is said to be subduplicate of the ratio of the first to the third. And if four magnitudes be continual proportionals, the ratio of the first to the second, or that of the second to the third, or that of the third to the fourth, is said to be subtriplicate of the ratio of the first to the fourth.

^h I think this definition is not clearly express'd; nay, I have often thought it to be scarcely sense; and therefore instead thereof I should rather say, *When four magnitudes are proportional, as the first is to the second, so is the third to the fourth; the first magnitude and the third, or the antecedents, as also the second and the fourth, or the consequents, of the two equal ratios, are called homologous or co-rational terms.* But this I submit to the learned.

ⁱ As let the magnitude A be to the magnitude B, as the magnitude C is to the magnitude D. Then alternately, or by permutation [see prop. 16.] it will be as A to C, so is B to D.

Alternate proportion cannot take place, unless the four proposed magnitudes be all of the same kind. For if a line A be to a line B, as a number C is to a number D, it would not be right to infer by alternation, that as the line A is to the number C, so is the line B to the number D, because there is no ratio

14. Inverse ratio is the assumption of the consequent as the antecedent to the antecedent taken as the consequent^k.

15. Composition of ratio is an assumption of the antecedent together with the consequent as one, to the same consequent^l.

16. Division of ratio is the assumption of the excess whereby the antecedent exceeds the consequent, to the same consequent^m.

17. Converse ratio, is the assumption of the antecedent to the excess whereby the antecedent exceeds the consequentⁿ.

18. Ratio of equality is, when there are several magnitudes in one rank or order, and as many others in another rank or order, comparing two to two, being in the same ratio, it shall be in the first rank as the first magnitude is to the last, so in the second rank shall the first magnitude be to the last. OR ELSE it is the assumption of the extremes by taking away the intermediate terms^o.

between a line and a number, as is evident from the fifth definition of this fifth book. But in the next following ways of arguing, *viz.* by inverse ratio, composition, division, and conversion, the two first magnitudes may be of one kind, and the two last of another, as is evident by the demonstrations of this book.

^k Let the magnitude A be to the magnitude B, as the magnitude C is to the magnitude D. Then inversely [by cor. 4. 5.] as B is to A, so is D to C.

^l As let the magnitude A be to the magnitude B, as the magnitude C is to the magnitude D. Then by composition [18. 5.] as A and B together is to B, so is C and D together to D.

^m As let the magnitude A be to the magnitude B, as the magnitude C is to the magnitude D. Then by division [17. 5.] as the difference between A and B is to B, so is the difference between C and D to D.

ⁿ As let the magnitude A be to the magnitude B, as the magnitude C is to the magnitude D. Then conversely [by cor. 19. 5.] as A is to the difference between A and B, so is C to the difference between C and D.

^o As let there be three magnitudes A, B, C in one rank or order, and three magnitudes D, E, F in another order. And let A be to B, as D is to E, and B be to C, as E is to F. Then by equality [by 22. 5.] will it be as A is to C, so is D to F.

19. Ordinate proportion is, when it shall be as an antecedent to a consequent, so is an antecedent to a consequent; and as a consequent is to some other magnitude, so is a consequent to some other magnitude r .

20. Inordinate or perturbate proportion, is when there are three magnitudes, and the same number of others, it shall be in the first magnitudes as an antecedent is to a consequent; so in the second magnitude is an antecedent to a consequent: But as in the first magnitudes a consequent is to some other magnitude, so in the second magnitudes is some other magnitude to the antecedent q .

r As let the magnitude A be to the magnitude B , as the magnitude C is to the magnitude D ; and again let one of the consequents B be to some other magnitude C , as the other consequent E is to some other magnitude F . Then is this ordinate proportionality; and it shall be true [by 22. 5.] that A is to C , as D is to F . I take this definition to be almost useless; it being in a manner contained in the last.

q As let the magnitude A be to the magnitude B , as the magnitude E is to the magnitude F . And again, as in the first magnitudes, the consequent B is to some other magnitude C ; so in the second magnitudes is some other magnitude D to the antecedent magnitude E . This sort of proportion is called inordinate or perturbate, because the same order is not kept in the proportion of the magnitudes. And it will be [by 23. 5.] as A is to C , so is D to F .

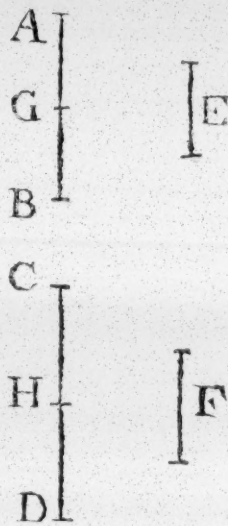
PROPOSITION I. THEOREM.

If there be how many soever magnitudes equimultiples of as many other magnitudes, each of each; the same multiple one magnitude is of one, shall all be of all^r.

Let any number of magnitudes $A B$, $C D$ be equimultiples of the same number of magnitudes E , F , each of each. I say $A B$ and $C D$ is the same multiple of E and F , as $A B$ is of E .

For because $A B$ is the same multiple of E , as $C D$ is of F ; as many magnitudes as there are in $A B$ equal to E , so many

many will there be in CD equal to F : Divide AB into parts equal to E , viz. AG, GB , and CD into parts equal to F , viz. CH, HD : Then the multitude of the parts CH, HD of the one is equal to the multitude of the parts AG, GB of the other. And because AG is equal to E , and CH to F ; AG, CH will be equal to E, F . By the same reason GB is equal to E , and HD to F : Therefore GB, HD are equal to E, F : Wherefore there are as many parts in AB equal to E , as there are in AB, CD equal to E, F : Therefore AB, CD will be the same multiple of E, F , as AB is of E .



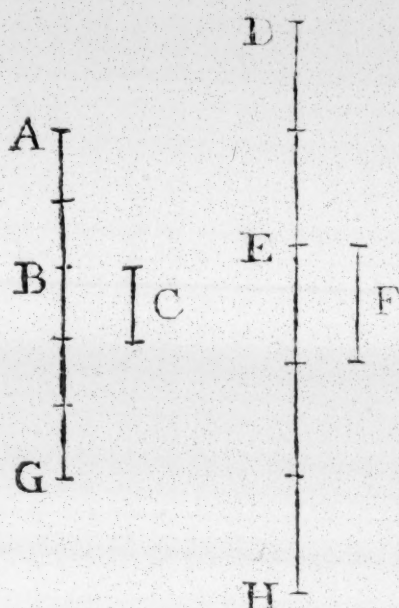
Therefore if there be how many soever magnitudes equimultiples of as many other magnitudes, each of each: the same multiple one magnitude is of one, all shall be of all. Which was to be demonstrated.

* The fifth and sixth propositions of the seventh book of Euclid answer to this in numbers. And what is here pronounced of multiple ratio only, is afterwards demonstrated universally at prop. 12. of any proportion, rational or irrational whatsoever. And this proposition was first put down to prove others, which were necessary towards the demonstration of the twelfth proposition.

PROP. II. THEOR.

If the first magnitude be the same multiple of the second, as the third is of the fourth, and there be a fifth the same multiple of the second, as a sixth magnitude is of the fourth; then the first and fifth taken together will be the same multiple of the second, as the third and sixth are of the fourth.

For let the first magnitude AB be the same multiple of the second C , as the third DE is of the fourth F . And let a fifth magnitude BG be the same multiple of the second C , as a sixth EH is of the fourth F : I say AG the first



and fifth taken together will be the same multiple of the second c , as DH the third and sixth taken together is of the fourth F .

For because AB is the same multiple of c , as DE is of F , there will be as many magnitudes in AB equal to c , as there are in DE equal to F . And by the same reason as many as are in BG equal to c , so many there are in EH equal to F . Therefore as many as there are in the whole AG , equal to c , so many there will be in the

whole DH equal to F : Therefore DH is the same multiple of F , as AG is of c : Wherefore AG the first and fifth taken together, is the same multiple of the second c , as DH the third and sixth, is of the fourth F .

Wherefore, if the first magnitude be the same multiple of the second, as the third is of the fourth; and a fifth magnitude be the same multiple of the second, as a sixth is of the fourth: Then the first and fifth taken together will be the same multiple of the second, as the third and sixth is of the fourth.

• This proposition supposes the fifth and sixth magnitudes to be equimultiples of the second and fourth. But the proposition would hold the same, if, the fifth were only equal to the second, and the sixth to the fourth.

PROP. III. THEOR.

If the first magnitude be the same multiple of the second as the third is of the fourth, and any equimultiples of the first and third be taken, then by equality, each of the assumed equimultiples will be an equimultiple of each, the one of the second, and the other of the fourth magnitude^t.

For let the first magnitude A be the same multiple of the second B , as the third C is of the fourth D ; and take

EF, GH equimultiples of A, C : I say EF is the same multiple of B, as GH is of D.

For because EF is the same multiple of A, as GH is of C : there will be as many magnitudes in GH equal to C, as there are in EF equal to A.

Divide EF into the magnitudes

EK, KF equal to A ; and GH into

the magnitudes GL, LH equal

to C : Then will the multitude of

EK, KF be equal to the multitude

of GL, LH. And because A is

the same multiple of B, as C is of

D, and EK is equal to A, and GL

equal to C ; EK will be the same

multiple of B, as GL is of D. By

the same reason KF is the same

multiple of B, as LH is of D. Then because the first EK

is the same multiple of the second B, as the third GL is of

the fourth D ; and the fifth KF is the same multiple of the

second B, as the sixth LH is of the fourth D : the magni-

tude EF made up of the first and fifth will [by 2. 5.] be

the same multiple of the second B, as GH the third and

sixth is of the fourth D.

If therefore the first magnitude be the same multiple of the second, as the third is of the fourth, and any equimultiples of the first and third be taken ; then, by equality, each of the assumed magnitudes will be an equimultiple of each, the one of the second, and the other of the fourth magnitude. Which was to be demonstrated.

* This is demonstrated universally at prop. 22. it being here only proposed in multiple ratio ; and that for the sake of demonstrating the fourth proposition.

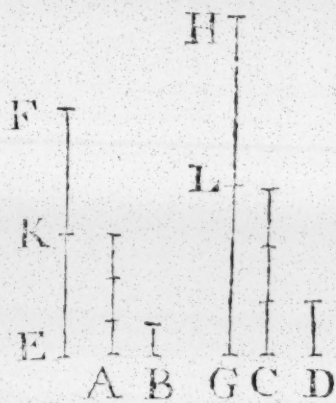
PROP. IV. THEOR.

If the first magnitude has the same ratio to the second, as the third has to the fourth ; then will any equimultiples of the first and third, have the same ratio to any equimultiples of the second and fourth, when compared to one another.

For let the first magnitude A have the same ratio to the second B, as the third C has to the fourth D. Take E, F

P 4

any



any equimultiples of A, C, and G, H, any other equimultiples of B, D: I say E will be to G, as F is to H.



For let κ , L be equimultiples of E, F, and M, N equimultiples of G, H.

Then because E is the same multiple of A, as F is of C, and κ , L are equimultiples of E, F: [by 3. 5.] κ will be the same multiple of A, as L is of C. By the same reason M will be the same multiple of B, as N is of D. And because as A is to B, so is C to D, and κ , L are taken any equimultiples of A, C and M, N any other equimultiples of B, D: If κ exceeds M; L [by 5. def. 5.] will exceed N, if κ be equal to M, L will be equal to N; if κ be less than M, L will be less than N; and κ , L are equimultiples of E, F: But M, N any other equimultiples of G, H: Therefore [by 5. def. 5.] as E is to G, so will F be to H.

Wherefore if the first magnitude has the same ratio to the second, as the third has to the fourth; then will any equimultiples of the first and third, have the same ratio to any equimultiples of the second and fourth, when compared to one another. Which was to be demonstrated.

Corollary. Therefore because it has been demonstrated, if κ exceeds M, L will also exceed N; if the one be equal to the one, the other will be equal to the other: if less, less: it is manifest if M exceeds κ , N will exceed L : if M be equal to κ , N will be equal to L : if less, less: Therefore as G is to E, so is H to F. Wherefore from hence it is manifest, that if four magnitudes be proportionals, they will be proportionals inversely.

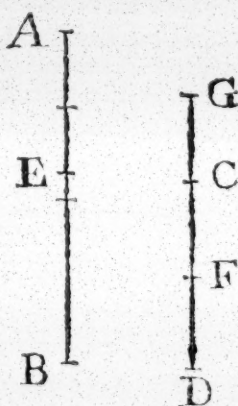
PROP. V. THEOR.

If one magnitude be the same multiple of another as a part taken away from the one is of a part taken away from the other: then shall the part remaining of the one be the same multiple of the part remaining of the other, as one of the whole magnitudes is of the other^v.

For let the magnitude AB be the same multiple of the magnitude CD , as the part AE , taken away of the one is of the part CF taken away of the other: I say the part EB remaining of the one, will be the same multiple of the part FD remaining of the other, as the whole magnitude AB is of the whole magnitude CD .

For make EB the same multiple of CG , as AE is of CF .

Then because [by I. 5.] AE is the same multiple of CF as AB is of GF , and AE is the same multiple of CF , as AB is of CD ; AB is the same multiple of GF , CD : Wherefore GF is equal to CD . Take away CF , which is common, from both: Then the remainder GC is equal to the remainder DF . And because AE is the same multiple of CF , as EB is of GC , and GC is equal to DF ; AE will be the same multiple of CF , as EB is of



FD . But AE is put the same multiple of CF , as AB is of CD : Therefore EB is the same multiple of FD , as AB is of CD : Wherefore the remainder EB will be the same multiple of the remainder FD , as the whole AB is of the whole CD .

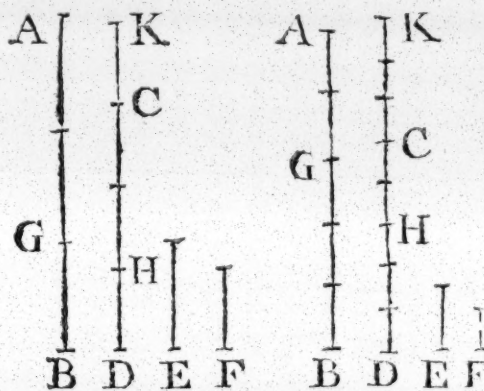
Therefore if one magnitude be the same multiple of another as a part taken away from the one, is of a part taken away from the other; then shall the part remaining of the one be the same multiple of the part remaining of the other, as one of the whole magnitudes is of the other. Which was to be demonstrated.

^v This proposition, proposed here only in multiple proportion, is universally demonstrated at prop. 19. and the seventh and eighth propositions of the seventh book answer to it in numbers.

PROP. VI. THEOR.

If two magnitudes be equimultiples of two other magnitudes, and some parts taken away from them, be equimultiples of the same magnitudes; then the remainders shall either be equal to the same magnitudes, or equimultiples of them^w.

For let two magnitudes AB, CD be equimultiples of two other magnitudes E, F , and the parts AG, CH taken away from them be some equimultiples of these other magnitudes E, F : I say the remainders GB, HD are either equal to E, F , or equimultiples of them.



For first let GB be equal to E : I say HD is equal to F : for put CK equal to F .

Then because AG is the same multiple of E , as CH is of F , and GB is equal to E , and CK to F ; AB [by 2. 5.] will be the same multiple of E , as KH is of F . But AB is put the same multiple of E , as CD is of F : Therefore KH is the same multiple of F as CD is of F : Wherefore because KH, CD are each an equimultiple of F ; KH will be equal to CD . Take away the common part CH ; and then the remainder KC is equal to the remainder HD . But KC is equal to F : Therefore HD is equal to F . If therefore GB be equal to E ; HD will be equal to F .

In like manner we demonstrate, whatever multiple GB is of E [as in the second figure] the same will HD be of F .

Therefore if two magnitudes be equimultiples of two other magnitudes, and some parts taken away from them be equimultiples of the same magnitudes: then the remainders will be either equal to the same magnitudes or equimultiples of them. Which was to be demonstrated.

^w This proposition being only in multiple proportion, is universally demonstrated at prop. 24. of all kinds of proportion.

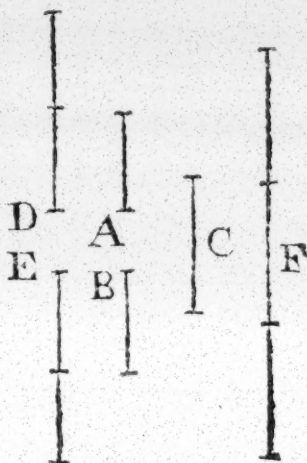
PROP. VII. THEOR.

Equal magnitudes have the same ratio to the same magnitude, and the same magnitude has the same ratio to equal magnitudes^{}.*

Let A, B be equal magnitudes, and c any magnitude whatsoever: I say each of these magnitudes A, B have the same ratio to c : and also c has the same ratio to A , or B .

For take D, E equimultiples of A and B , and take any other multiple F of c .

Then because D is the same multiple of A , as E is of B , and A is equal to B ; D will be equal to E . But F is any other multiple of c : Therefore if D exceeds F , E will exceed F too; and if it be equal to F , E will be equal to F . If less, less. But D, E are equimultiples of A, B , and F is any multiple whatsoever of c : Therefore [by 5. def. 5.] A will be to c , as B is to c .



I say moreover that c has the same ratio to A or B .

For the construction remaining the same, we demonstrate in like manner that D is equal to E . But F is any other magnitude. Therefore if F exceeds D , it will also exceed E ; if F be equal to D , it will be equal to E ; if less, less. But F is a multiple of c , and D, E are any equimultiples of A, B : Therefore [by 5. def. 5.] as c is to A , so will c be to B .

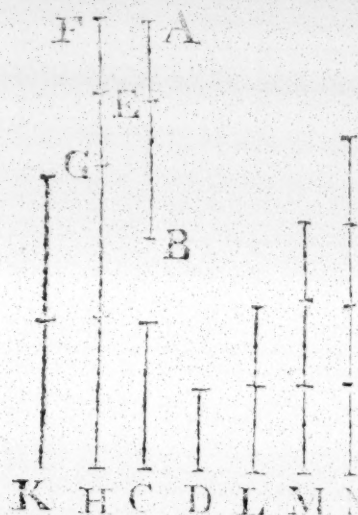
Therefore equal magnitudes have the same ratio to the same magnitude, and the same magnitude has the same ratio to equal magnitudes. Which was to be demonstrated.

^{*} This seventh, with the eighth, ninth, tenth, eleventh, and twelfth propositions following, are taken by some to be mere axioms requiring no demonstration at all. They are indeed very evident in numbers. But since they are applicable to all magnitudes in general, lines, planes, solids, commensurable, and incommensurable, Euclid could not but demonstrate them.

PROP. VIII. THEOR.

The greater of [two] unequal magnitudes has a greater ratio to the same magnitude, than the lesser; and the same magnitude to the lesser of [two] unequal magnitudes has a greater ratio than it has to the greater [of these two] magnitudes.

Let AB, C , be two unequal magnitudes, and let AB be greater than C . Also let D be any other magnitude whatsoever. I say AB has a greater ratio to D , than C has to D : and D has a greater ratio to C , than it has to AB .



For because AB is greater than C , make [by 3. 1.] BE equal to C ; then the lesser of these two magnitudes AE, BE being multiplied, will at length exceed D [by 4. def. 5.]. First let AE be less than EB , and multiply AE so often till it exceeds D : Let FG be this multiple of AE , which is greater than D . Also make GH the same multiple of EB , and K of C , as FG is of AE . And take L double to

D , M triple to it, and so forwards greater by one, until the magnitude taken be a multiple of D , and [in the first place] greater than K . Let N be this magnitude, being four times the magnitude D , and [in the first place] greater than K .

Then because K [in the first place] is less than N ; K will not be lesser than M . And since FG is the same multiple of AE as HG is of EB ; [by 1. 5.] FG will be the same multiple of AE as FH is of AB . But FG is the same multiple of AE as K is of C : Therefore FH is the same multiple of AB as K is of C . And so FH, K are equimultiples of AB , and C . Again, because GH is the same multiple of EB as K is of C , and EB is equal to C ; GH will be equal to K . But K is not less than M . Therefore GH is not less than M . But [by constr.] FG is greater than D : Therefore the whole FH will be greater than

D, M together. But D, M together are equal to N; wherefore FH exceeds N; but K does not exceed N; and FH, K are equimultiples of AB, C; and N is any other multiple of D: Therefore [by 7. def. 5.] AB has a greater ratio to D, than C has to D.

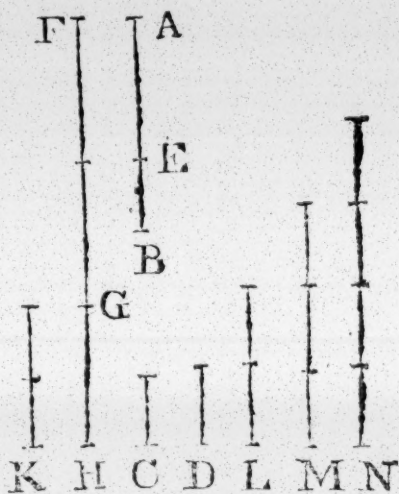
I say moreover that the ratio of D to C is greater than the ratio of D to AB.

For the same construction remaining, we demonstrate in the like manner, that N exceeds K, but does not exceed FH. But N is a multiple of D, and FH, K any other equimultiples of AB, C. Therefore [by 7. def. 5.] the ratio of D to C is greater than the ratio of D to AB.

Now let AE be greater than EB: Then [by 4. def. 5.] the lesser magnitude EB being multiplied, will at length become greater than D. Let it be multiplied, and let GH the multiple of EB be greater than D: And make FG the same multiple of AE, and K of C, as GH is of EB. Then by the like reason as before, we demonstrate that FH, K are equimultiples of AB, C. And likewise take N a multiple of D, [in the first place] greater than FG: Therefore again FG is not less than M. But GH is greater than D. Therefore the whole FH exceeds D and M together, that is N. But K does not exceed N, because FG, which is greater than HG, that is, than K does not exceed N. And after the like manner as before we finish the demonstration.

Therefore the greater of [two] unequal magnitudes has a greater ratio to the same magnitude than the lesser; and the same magnitude to the lesser of [two] unequal magnitudes, has a greater ratio than it has to the greater of those [two] magnitudes. Which was to be demonstrated.

PROP.



PROP. IX. THEOR.

Magnitudes that have the same ratio to the same magnitude, are equal to one another: and those magnitudes to which the same magnitude has the same ratio, are also equal to one another.

For let each of the magnitudes A and B have the same ratio to the same magnitude c: I say A is equal to B.

B

A

For if it were not equal, A, and B would not [by 8. 5.] have the same ratio to c.

But it has. Therefore A is equal to B.

Again, let c have the same ratio both to A, B. I say A is equal to B.

C

For if it were not equal, c would not have the same ratio both to A and B. But it has. Therefore A is equal to B.

Wherefore magnitudes that have the same ratio to the same magnitude, are equal to one another: and those magnitudes to which the same magnitude has the same ratio, are also equal to one another. Which was to be demonstrated.

PROP. X. THEOR.

That magnitude of those magnitudes which have a ratio to the same magnitude, is the greater, which has the greater ratio: and that magnitude to which the same magnitude has the greater ratio, is the lesser.

For let A have a greater ratio to c, than B has to c: I say A is greater than B.

B

A

C

For if it be not greater, it is either equal to it, or less than it: But A is not equal to B, for then [by 7. 5.] A and B would each have the same ratio to c. But they have not the same ratio. Therefore A is not equal to B. Nor is A less than B, for then [by 8. 5.] A would have a less ratio to c than B has to c. But it has not a less. Wherefore A is not less than B: it is therefore greater.

Again,

Again let c have a greater ratio to B , than it has to A : I say B is less than A .

For if it be not less, it is either equal to it, or greater. But B is not equal to A , for then c [by 7. 5.] would have the same ratio to A , as to B . But it has not: Therefore A is not equal to B , nor is B greater than A ; for then [by 8. 5.] c would have a less ratio to B than to A . But it has not: Therefore B is not greater than A : But it has been proved not to be equal to it: Consequently B will be less than A .

Therefore that magnitude of those magnitudes which have a ratio to the same magnitude, is the greater, which has the greater ratio: and that magnitude to which the same magnitude has the greater ratio, is the lesser.

PROP. XI. THEOR.

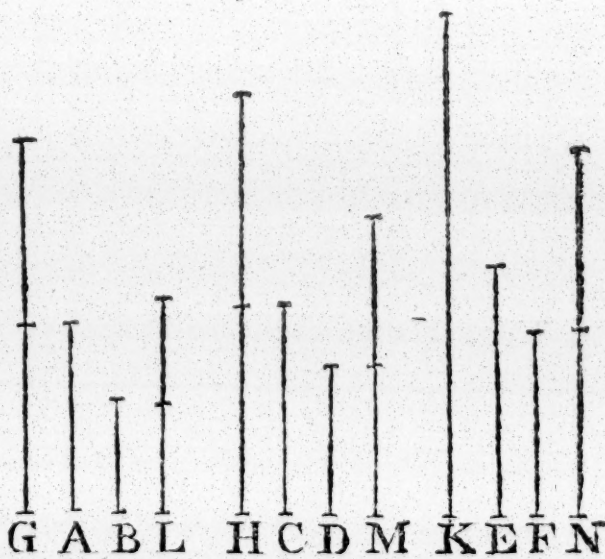
Those ratios that are the same to the same ratio, are the same between themselves.

For let A be to B , as c is to D , and as c is to D , so is E to F : I say as A is to B , so is E to F .

For take G, H, K equimultiples of A, c, E , and any other equimultiples L, M, N of B, D, F .

Then because it is as A to B , so is c to D , and G, H

are equimultiples of A, c , and L, M any other equimultiples of B, D : If G exceeds L , [by 5. def. 5.] H will exceed M : if, the one be equal, the other will be equal; if less, less. Again, because it is as c to D , so



is E to F , and H, K are taken equimultiples of c, E ; also M, N any other equimultiples of D, F : If H exceeds M ; K will exceed N . If equal, equal; but if less, less. But if H exceeds

ceeds

ceeds M , G will exceed L : If it be equal, equal; if less, less. Wherefore if G exceeds L , K will exceed N : if equal, equal: if less, less. And G , K are equimultiples of A , E , and L , N any others of B , F : Therefore [by 5. def. 5.] as A is to B , so will E be to F .

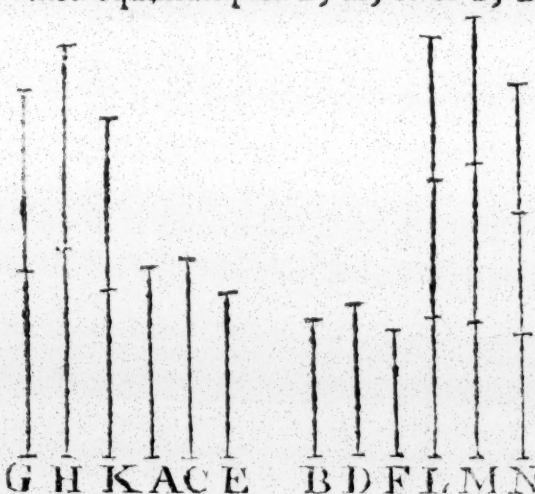
Wherefore those ratios that are the same to the same ratio, are the same between themselves. Which was to be demonstrated.

PROP. XII. THEOR.

If any magnitudes soever be proportional; as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.

Let any magnitudes soever A , B , C , D , E , F be proportional; and let A be to B , as C is to D , and as E is to F : I say as A is to B , so is A , C , E , to B , D , F .

For take the equimultiples G , H , K of A , C , E , and any other equimultiples L , M , N of B , D , F .



Then because as A is to B , so is C to D , and E to F : And there are taken the equimultiples G , H , K of A , C , E , and any other equimultiples L , M , N of B , D , F : If G exceeds L [by 5. def. 5.] H will exceed M , and K will exceed N : If the one be equal, the other will be equal: if less, less. Wherefore also if G exceeds L ; G , H , K will exceed L , M , N ; if equal, equal; and if less, less. And G , and C , H , K are equimultiples of A and A , C , E : Because [by 1. 5.] if there be any number of magnitudes soever equimultiples of an equal number of magnitudes, each the same multiple of each, the same multiple that one of the magnitudes is of one of the other, will all the magnitudes be of all the others. By the same reason L and L , M , N are equimultiples of B , and

B , D , F :

B, D, F : Therefore [by 5. def. 5.] as A is to B, so is A, C, E, to B, D, F.

Wherefore if any magnitudes soever be proportional : as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents. Which was to be demonstrated.

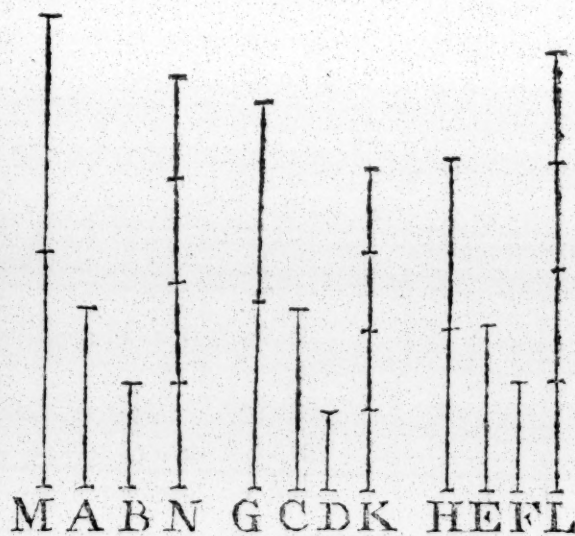
The twelfth proposition of the seventh book of Euclid answers to this in numbers.

PROP. XIII. THEOR.

If the first magnitude has the same ratio to the second, as the third to the fourth ; but the third to the fourth has a greater ratio than the fifth to the sixth ; then will the first have a greater ratio to the second, than the fifth has to the sixth.

For let the first magnitude A have the same ratio to the second B, that the third C, has to the fourth D ; and let the third C have a greater ratio to the fourth D, than the fifth E has to the sixth F : I say the first A will have a greater ratio to the second B, than the fifth E has to the sixth F.

For because the ratio of C to D is greater than the ratio of E to F, there are [by 7. def. 5.] some equimultiples of C, E, and others of D, F ; where the multiple of C exceeds the multiple of D, but the multiple of E does not exceed the multiple of F. Take



these equimultiples. And let G, H be equimultiples of C, E, and K, L other equimultiples of D, F, such that G exceeds K, but H does not exceed L : and the same multiple that G is of C, the same let M be of A ; and the same that K is of D, let N be of B.

Q

Then

Then because A is to B, as C is to D, and there are taken the equimultiples M, G of A, C, and other equimultiples N, K of B, D : [by 5. def. 5.] If M exceeds N, G will exceed K : if M be equal to N, G will be equal to K : if less, less : But G exceeds K ; therefore M will exceed N : But H does not exceed L ; and M, H are equimultiples of A, E, and N, L some other equimultiples of B, F : Therefore [by 7. def. 5.] A will have a greater ratio to B than E has to F.

If therefore the first magnitude has the same ratio to the second, as the third to the fourth ; and the third to the fourth has a greater ratio than the fifth to the sixth : then the first to the second will have a greater ratio than the fifth has to the sixth. Which was to be demonstrated.

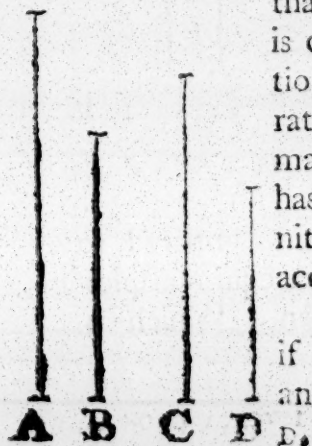
PROP. XIV. THEOR.

If the first magnitude has the same ratio to the second, as the third has to the fourth ; and the first be greater than the third : then will the second be greater than the fourth. If the first be equal to the third, the second will be equal to the fourth. If the first be less than the third, the second will be less than the fourth.

For let the first magnitude A have the same ratio to the second B, as the third C has to the fourth D, and let A be greater than C : I say B is also greater than D.

For because A is greater than C, and B is any other magnitude ; [by 8. 5.] A will have a greater ratio to B, than C has to B. But as A is to B, so is C to D : Therefore [by 13. 5.] the ratio of C to D, will be greater than the ratio of C to B. But [by 10. 5.] that magnitude to which the same magnitude has the greater ratio, is the lesser magnitude : wherefore D is less than B : and accordingly B will be greater than D.

After the like manner we demonstrate if A be equal to C, that B is equal to D : and if A be less than C, B is less than D.



Therefore

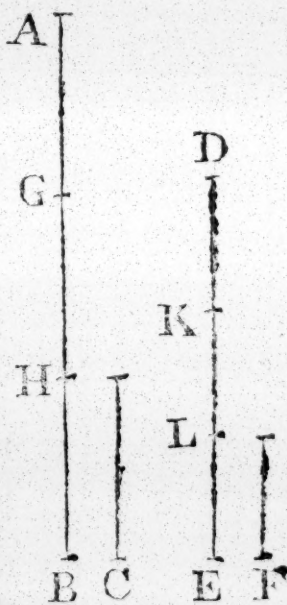
Therefore if the first magnitude has the same ratio to the second, as the third has to the fourth, and the first be greater than the third: then will the second be greater than the fourth; if the first be equal to the third, the second will be equal to the fourth: and if the first be less than the third, the second will be less than the fourth. Which was to be demonstrated.

PROP. XV. THEOR.

The parts of magnitudes compared to one another, have the same ratio as their like equimultiples have to one another when answerably taken².

For let AB be the same multiple of C , that DE is of F : I say C is to F , as AB is to DE .

For because AB is the same multiple of C , that DE is of F : there will be as many magnitudes in AB equal to C , as there are in DE , equal to F . Divide AB into magnitudes equal to C , which let be AG, GH, HB ; and divide DE into the magnitudes DK, KL, LE each equal to F . Then the multitude of AG, GH, HB will be equal to the multitude of DK, KL, LE . And because AG, GH, HB are equal to one another, as also DK, KL, LE equal to one another: [by 7. 5.] AG will be to DK , as GH is to KL , and as HB is to LE : But [by 12. 5.] as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents. Therefore AG is to DK , as AB is to DE . But AG is equal to C , and DK equal to F . Therefore as C is to F , so will AB be to DE .



Wherefore the parts of magnitudes compared to one another have the same ratio as their like equimultiples have to one another when answerably taken. Which was to be demonstrated.

² The seventeenth proposition of the seventh book answers to this proposition.

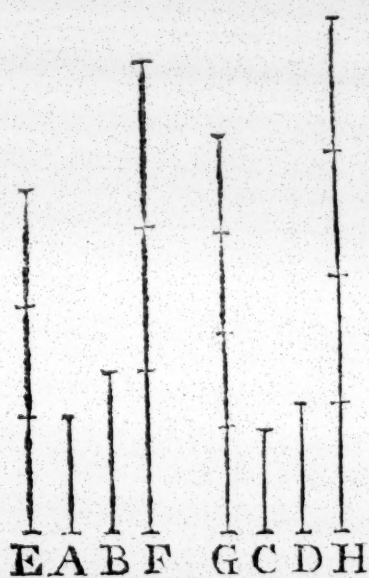
PROP. XVI. THEOR.

*If four * magnitudes be proportional, they will be alternately proportional.*

Let the four magnitudes A, B, C, D be proportional; viz. let A be to B , as C is to D : I say they are also alternately proportional; that is, as A is to C , so is B to D .

For take any equimultiples E, F of A, B , and any other equimultiples G, H of C, D .

Then because E is the same multiple of A , that F is of B , and the parts of magnitudes compared to one another



[by 15. 5.] have the same ratio, that their equimultiples have to one another; A will be to B , as E is to F . But as A is to B , so is C to D : Wherefore [by 11. 5.] as C is to D , so is E to F . Again, because G, H are equimultiples of C and D ; C will be to D , as G is to H . But as C is to D , so is E to F : Wherefore [by 11. 5.] as E is to F , so is G to H . But if four magnitudes be proportional, and the first be greater than the third; [by 14. 5.] the second will be greater than the fourth: If the first be

equal to the third, the second will be equal to the fourth: and if the first be less than the third, the second will be less than the fourth: If therefore E exceeds G ; F will also exceed H ; if E be equal to G , F will be equal to H , and if E be less than G , F will be less than H . But E, F are any equimultiples of A, B , and G, H any other equimultiples of C, D : Therefore [by 5. def. 5.] as A is to C , so will B be to D .

If therefore four magnitudes be proportional; they will be also alternately proportional. Which was to be demonstrated.

* Of the same kind, either lines, superficies, or solids. The thirteenth proposition of the seventh book answers to this in numbers.

PROP.

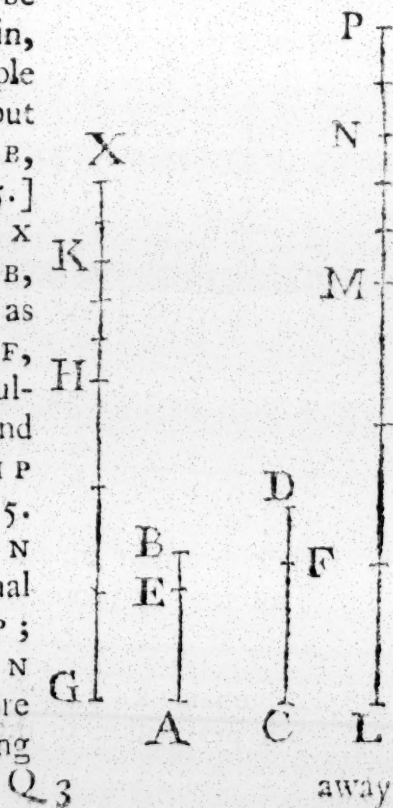
PROP. XVII. THEOR.

If magnitudes when compounded, be proportional; they will be also proportional when divided.

Let the compounded magnitudes AB, BE, CD, DF be proportional, and let AB be to BE , as CD is to DF : I say they will be also proportional when they are divided; that is, as AE is to EB , so is CF to FD .

For take any equimultiples GH, HK, LM, MN of AE, EB, CF, FD , and any other equimultiples KX, NP of EB and FD .

Then because GH is the same multiple of AE , that HK is of EB ; [by 1. 5.] GH will be the same multiple of AE , that GK is of AB . But GH is the same multiple of AE , that LM is of CF : Therefore GK will be the same multiple of AB that LM is of CF . Again, because LM is the same multiple of CF , that MN is of FD ; LM will be the same multiple of CF , that LN is of CD . But LM was the same multiple of CF , that GK is of AB : Therefore GK is the same multiple of AB , that LN is of CD : Wherefore GK, LN will be equimultiples of AB, CD . Again, because HK is the same multiple of EB , that MN is of FD ; but KX is the same multiple of EB , that NP is of FD : Also [by 2. 5.] the compounded magnitude HX will be the same multiple of EB , that MP is of FD . But since as AB is to BE , so is CD to DF , and there are taken any equimultiples GK, LN of AB, CD , and any other equimultiples HX, MP of EB, FD : Therefore [by 5. def. 5.] if GK exceeds HX ; LN will exceed MP : if GK be equal to HX , LN will be equal to MP ; and if GK be less than HX ; LN will be less than MP . Therefore let GK exceed HX ; and taking



away HK which is common, GH will exceed KX . But if GK exceeds HX ; LN will exceed MP : so that LN exceeds MP ; and MN common to both being taken away, LM will exceed NP : wherefore if GH exceeds KX ; LM will exceed NP . In like manner we demonstrate that if GH be equal to KX , LM is equal to NP ; and if GH be less than KX ; LM will also be less than NP . But GH , LM are any equimultiples of AE , CF , and KX , NP any other equimultiples of EB , FD : Therefore [by 5. def. 5.] as AE is to EB , so will CF be to FD .

If therefore magnitudes when compounded are proportional; they will be also proportional when they are divided. Which was to be demonstrated.

PROP. XVIII. THEOR.

If magnitudes when divided be proportional, they will also be proportional when they are compounded.

Let the divided magnitudes AE , EB , CF , FD be proportionals, viz. as AE is to EB , so is CF to FD : I say they are also proportional when compounded; that is, as AB is to BE , so is CD to FD .

For if it be not as AB is to BE , so is CD to FD ; it will be as AB is to BE , so is CD , to some magnitude either lesser than FD , or greater than it.

First let it be to some magnitude DG which is less than FD . Then because it is as AB is to BE , so is CD to DG , these compounded magnitudes are proportional: Therefore [by 17. 5.] they will be also proportional when they are divided: Wherefore it is as AE to EB , so is CG to GD . But [by supposition] as AE is to EB , so is CF to FD : Wherefore [by 11. 5.] as CG is to GD , so is CF to FD . But the first magnitude CG is greater than the third CF : Therefore [by 14. 5.] the second GD will be greater than the fourth FD . But it is less too; which is absurd. Therefore it is not as AB is to BE , so is CD to a magnitude that is less than FD . After the same manner we demonstrate that it cannot be as AB

is to BE, so is CD to a magnitude greater than FD. Therefore it is as AB is to BE, so is CD to FD.

Wherefore if magnitudes when divided be proportional, they will also be proportional when they are compounded. Which was to be demonstrated.

PROP. XIX. THEOR.

If one magnitude be to another, as a part taken away of the one is to a part taken away of the other; then shall the remaining part of the one magnitude be to the remaining part of the other, as one of the magnitudes is to the other.

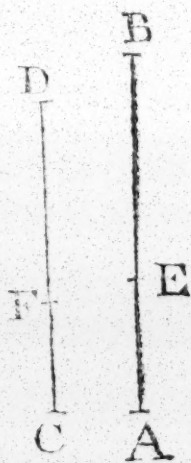
For let it be as one magnitude AB is to another CD, so is the part AE of the one to the part CF of the other: Then I say the remainder EB of the one is to the remainder FD of the other, as the magnitude AB is to the magnitude CD.

For because it is as the whole AB is to the whole CD, so is AE to CF; and alternately [by 16. 5.] as BA is to AE, so is DC to CF. And because if magnitudes when compounded are proportionals, they will be [by 17. 5.] proportionals when divided: Therefore as BE is to EA, so will DF be to CF; and again, alternately, as BE is to DF, so is EA to FC. But as AE is to CF, so [by supposition] is AB to CD. Therefore [by 11. 5.] the remaining part EB will be to the remaining part FD, as the whole AB is to the whole CD.

Wherefore if one magnitude be to another, as a part of the one is to a part of the other; then shall the remaining part of one of the magnitudes be to the remaining part of the other, as one magnitude is to the other. Which was to be demonstrated.

Corollary. * And because it is demonstrated [by 16. 5.] as AB is to CD, so is EB to FD: if it be alternately as

* This corollary is very corrupt; nor can it be corrected from any of the ancient copies: I have therefore altered the



AB is to BE , so is CD to DF , that is, compounded magnitudes proportional. But it has been demonstrated [by 16. and 19. 5.] that AB is to AE , as CD is to CF , which [by 17. def. 5.] is converse ratio. Therefore it is manifest from hence if compound magnitudes be proportional, that they will also be proportional conversely.

PROP. XX. THEOR.

If there be three magnitudes and others equal to them in number, which taken two and two are in the same ratio. And by equality the first is greater than the third: then will the fourth be greater than the sixth: If the first be equal to the third, the fourth will be equal to the sixth; and if the first be less than the third, the fourth will be less than the sixth.

Let there be three magnitudes A, B, C , and D, E, F any others equal to them in number, which taken two and two are in the same ratio, and let A be to B , as D is to E , and as B is to C , so is E to F , and by equality A is greater than C : I say also that D is greater than F : If A be equal to C , D will be equal to F : and if A be less than C , D will be less than F .



For because A is greater than C , and B is any other magnitude, and [by 8. 5.] the greater of two magnitudes has a greater ratio to the same magnitude than the lesser; the ratio of A to B , will be greater than the ratio of C to B . But as A is to B , so is D to E , and [by inversion] as C is to B , so is F to E : Therefore D has a greater ratio to E , than F has to E .

translation to preserve the sense. But the following is the most legitimate demonstration of converse ratio. If AB be to BE , as CD is to DF , it will be [by dividing] as AE is to BE , so is CF to DF , and [by inversion] as BE is to AE , so is DF to CF ; and [by compounding] it will be as AB is to AE , so is CD to CF , which is converse ratio.

But

But of two magnitudes which have a ratio to the same magnitude, that which has the greatest ratio is the greater magnitude [by 10. 5.]: Therefore D is greater than F: After the like manner we demonstrate, that if A be equal to c, D will be equal to F: and if A be less than c, D will be less than F.

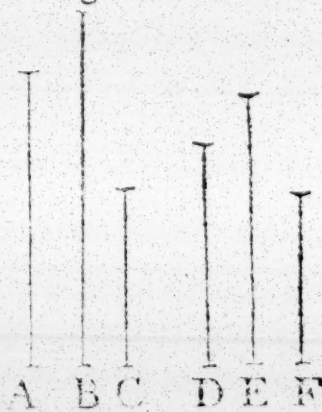
If therefore there be three magnitudes, and others equal to them in number, which taken two and two are in the same ratio, and by equality the first is greater than the third: then will the fourth be greater than the sixth: If the first be equal to the third, the fourth will be equal to the sixth: and if the first be less than the third, the fourth will be less than the sixth. Which was to be demonstrated.

PROP. XXI. THEOR.

If there be three magnitudes, and others equal to them in number, which taken two and two are in the same ratio; and let their proportion be perturbate, and if, by equality, the first be greater than the third: then will the fourth be greater than the sixth: if the first be equal to the third, the fourth will be equal to the sixth: and if the first be less than the third, the fourth will be less than the sixth.

Let there be three magnitudes A, B, C, and others D, E, F equal to them in number, which taken two and two are in the same ratio; and let their proportion be perturbate, viz. as A is to B, so is E to F, and as B is to C, so is D to E. And, by equality, A is greater than C: I say D is greater than F; if A be equal to C, D will be equal to F: and if A be less than C, D will be less than F.

For because A is greater than C, and B is another magnitude; [by 8. 5.] the ratio of A to B will be greater than the ratio of C to B. But as A is to B, so is E to F, and inversely as C is to B, so is E to D: Therefore E will have a greater ra-



tio to F than it has to D . But when the same magnitude has ratios to two others, that which has the greater ratio [by 10. 5.] is the lesser magnitude: Therefore F is less than D : and so D will be greater than F . In like manner we demonstrate if A be equal to C ; D will be equal to F ; and if A be less than C ; D will be less than F .

If therefore there be three magnitudes, and others equal to them in number, which taken two and two are in the same ratio; and their proportion be perturbate, and if by equality, the first be greater than the third; then will the fourth be greater than the sixth: If the first be equal to the third, the fourth will be equal to the sixth: and if the first be less than the third, the fourth will be less than the sixth. Which was to be demonstrated.

PROP. XXII. THEOR.

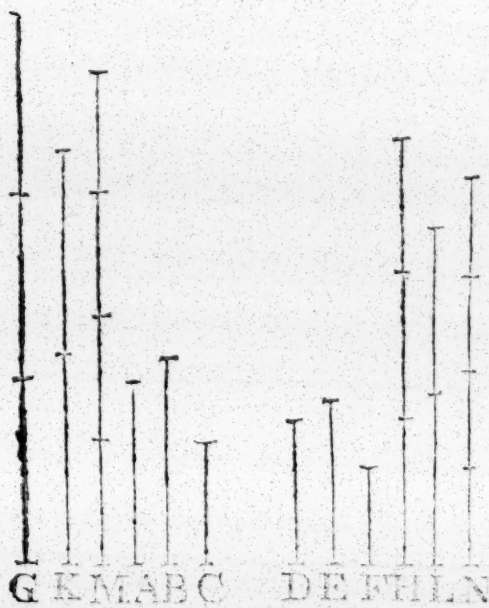
If there be any number soever of magnitudes and others equal to them in number, which taken two and two are in the same ratio; they will be also by equality in the same ratio.

Let there be any number soever of magnitudes A, B, C , and others D, E, F equal to them in number, which taken

two and two are in the same ratio, as A is to D , so is B to E , and as B is to C , so is E to F : I say, by equality, they shall be in the same ratio, as A is to C , so is D to F .

For take G, H , equimultiples of A, D , and K, L any other equimultiples of B, E ; and M, N other equimultiples of C, F .

Then because A is to D , as B is to E , and G ,



H , are taken equimultiples of A, D , and K, L any others of B, E ; [by 4. 5.] it will be as G is to K , so is H to L . Also by the same reason K will be to M , as L is to N .

And

And since there are three magnitudes G, K, M , and others H, L, N equal to them in number, which taken two and two are in the same ratio: Therefore by equality, [by 20. 5.] if G exceeds M ; H will exceed N ; if G be equal to M , H will be equal to N : If G be less than M , H will be less than N . And G, H are equimultiples of A, D , and M, N any others of C, F : Therefore as A is to C , so will [by 5. def. 5.] D be to F .

Wherefore if there be any number soever of magnitudes, and others equal to them in number which taken two and two are in the same ratio; they will be also by equality in the same ratio. Which was to be demonstrated.

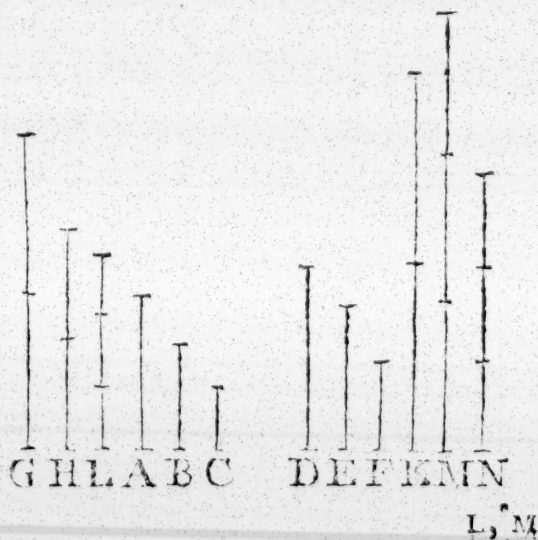
PROP. XXIII. THEOR.

If there be three magnitudes, and others equal to them in number, which taken two and two are in the same ratio; and their proportion be perturbate; then they will be in the same proportion by equality.

Let there be three magnitudes A, B, C , and others D, E, F equal to them in number, which taken two and two are in the same ratio; and let their proportion be perturbate; viz. as A is to B , so is E to F ; and as B is to C , so is D to E : I say as A is to C , so is D to F .

For take G, H, K equimultiples of A, B, D , and L, M, N any other equimultiples of C, E, F .

Then because G, H are equimultiples of A, B , and [by 15. 5.] the parts of magnitudes have the same ratio, as their equimultiples; it will be as A to B , so is G to H . And by the same reason as E is to F , so is M to N : But as A is to B , so is E to F . Therefore as G is to H , so [by 11. 5.] will M be to N . And because it is as B to C , so is D to E , and H, K are taken equimultiples of B, D , and



L, M any others of C, E; [by 15. 5.] it shall be as H is to L, so is K to M. But it has been proved that G is to H, as M is to N. Wherefore because there are three magnitudes G, H, L, and others K, M, N equal to them in number, which taken two and two are in the same ratio; and their proportion is perturbate, then by equality, [by 21. 5.] if G exceeds L; K will exceed N: if G be equal to L; K will be equal to N: and if G be less than L; K will also be less than N. But G, K are equimultiples of A, D, and L, N equimultiples of C, F: Therefore as A is to C, so will [by 5. def. 5.] D be to F.

Wherefore if there be three magnitudes, and others equal to them in number, which taken two and two are in the same ratio, and their proportion be perturbate: they will be, by equality, proportional. Which was to be demonstrated.

^a Euclid proposes only three magnitudes here; but they may be any number whatsoever, as well as they may in prop. 22. a geometrician seldom having occasion for any more. The fourteenth and twenty second propositions of the seventh book answer to these propositions in numbers.

PROP. XXIV. THEOR.

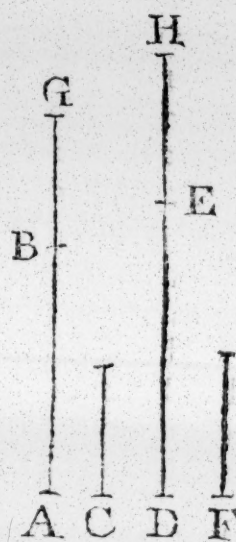
If the first magnitude has the same ratio to the second, as the third has to the fourth, and the fifth has the same ratio to the second as the sixth has to the fourth: then shall the magnitude compounded of the first and fifth have the same ratio to the second, as the magnitude compounded of the third and sixth has to the fourth.

For let the first magnitude AB have the same ratio to the second C, that the third DE has to the fourth F: and let the fifth BG have the same ratio to the second C, as the sixth EH, has to the fourth F: I say the magnitude AG compounded of the first and fifth, will have the same ratio to the second C, as DH the magnitude compounded of the third and sixth, has to the fourth F.

For because as BG is to C, so is EH to F; it will be inversely [by cor. 4. 5.] as C is to BG, so is F to EH. And because as AB is to C, so is DE to F, and as C is

to BG , so is F to EH ; by equality [by 22. 5.] AB will be to BG , as DE is to EH . But since when magnitudes divided are proportional, [by 18. 5.] they will be also proportional when they are compounded: it will be therefore as AG is to GB , so is DH to HE . But as GB is to C , so is EH to F : Therefore by equality [by 22. 5.] it will be as AG is to C , so is DH to F .

If therefore the first magnitude has the same ratio to the second, as the third has to the fourth, and the fifth has the same ratio to the second as the sixth has to the fourth: then shall the magnitude compounded of the first and fifth have the same ratio to the second, as the magnitude compounded of the third and sixth has to the fourth. Which was to be demonstrated.



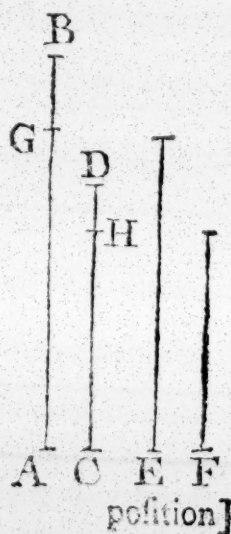
PROP. XXV. THEOR.

If four magnitudes be proportional, the greatest and least of them taken together will be greater than the two others taken together^b.

Let the four magnitudes AB , CD , E , F be proportional, and let it be as AB is to CD , so is E to F , and let AB be the greatest of them, and F the least: I say AB and F are greater than CD and E .

For make AG equal to E , and CH equal to F .

Then because AB is to CD , as E is to F , and AG is equal to E , and CH equal to F ; it shall be as AB to CD , so is AG to CH . And because one magnitude AB is to another CD , as the part AG of the one is to the part CH of the other; [by 19. 5.] the remaining part GB of the one will be to the remaining part HD of the other, as one magnitude AB is to the other CD . But [by sup-



position] AB is greater than CD ; therefore GB is greater than HD . But since AG is equal to E , and CH to F ; AG and F will be equal to CH and E . But if equal magnitudes be added to unequal ones, the wholes will be unequal: Wherefore since GB , HD are unequal, and GB is the greater, if AG and F be added to GB , and CH and E to HD ; AB and F will be greater than CD and E .

If therefore four magnitudes be proportional, the greatest and least of them taken together will be greater than the other two taken together. Which was to be demonstrated.

^b If the antecedent of one of the ratios be a maximum, the consequent of the other will be a minimum. And contrariwise, if the antecedent of one of the ratios be a minimum, the consequent of the other will be a maximum. But all the four magnitudes must be of the same kind, *viz.* lines, superficies, or solids, otherwise the proposition will not hold good: For the greatest and least put together cannot make one magnitude, no more than the other two can.

It is easy to demonstrate this theorem in right lines, by means of prop. 7. and 35. lib. iii. and prop. 16. lib. vi.

EUCLID'S

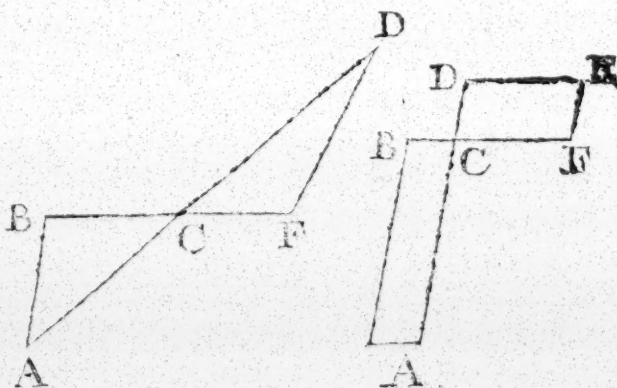
E U C L I D's E L E M E N T S. B O O K VI.

D E F I N I T I O N S.

1. **S**imilar right lined figures are those which have their several angles equal each to each, and the sides about the equal angles proportional.

2. Figures are reciprocal, when in each figure, the antecedents and consequents are the terms of ratios ^a.

^a In the triangles $A B C$, $C D F$ or the parallelograms $A B$, $D F$, if the side $B C$ of the one be to the side $C F$ of the other as back again is the side $C D$ of this other, to the side $A C$ of the former. Where



the first antecedent $B C$ and the last consequent $A C$ of the equal ratios $B C$ to $C F$, as $C D$ to $A C$, are both in one of the figures, and also the second antecedent $C D$ and the first consequent $C F$ are also both in the same figure, those triangles and parallelograms are said to be reciprocal figures.

Two figures are therefore said to be reciprocal, or their sides are said to be reciprocally proportional when their sides are so proportional, that there is one antecedent of one of two equal ratios, and the consequent of the other equal ratio, in each figure.

And as the sides about two equal angles of equal triangles

3. A right line is said to be cut or divided into mean and extreme ratio, when that line shall be to the greater segment, as the greater segment is to the lesser.

4. The altitude of any figure is the perpendicular drawn from the vertex [or top] to the base ^b.

5. A ratio is said to be compounded of ratios when the quantities of the ratios multiplied between themselves produce that ratio ^c.

are always reciprocally proportional, so will the sides about the equal angles of equiangular triangles, be always directly proportional. These distinctions of reciprocal and direct proportion are very useful, and ought therefore to be particularly regarded.

^b The altitude or height of a figure is the distance or shortest length from the bottom of it to the top; and the perpendicular rather measures that distance, than is the distance itself; and when figures have the same altitude, the perpendiculars drawn from their vertexes or tops upon their bases are equal.

^c As let there be any number of quantities generally represented by the letters A, B, C, D, E, the ratio of the extremes A and E is compounded of the ratios of A to B, of B to C, of C to D, and of D to E; that is, if A divided by B be the measure or quantity of the ratio of A to B; then will $\frac{A}{B}$ in form of a fraction, be the quantity of the ratio of A to B; $\frac{B}{C}$ the quantity of the ratio of B to C; $\frac{C}{D}$ the quantity of the ratio of C to D, and $\frac{D}{E}$ the quantity of the ratio of D to E. And the product of the multiplication of all these quantities, according to the rules of multiplication of fractions, that is, multiplying all the numerators A, B, C, D together, and dividing them by all the denominators B, C, D, E multiplied together (or A into B into C into D divided by B into C, into D into E) will be the compound ratio of A into E. For the product B into C into D being both in the numerator, and the denominator of that fraction $\frac{A}{B}$ into $\frac{B}{C}$ into $\frac{C}{D}$ into $\frac{D}{E}$ may be taken away from both. And then will $\frac{A}{B}$ into $\frac{B}{C}$ into $\frac{C}{D}$ into $\frac{D}{E}$ become equal to $\frac{A}{E}$. Or A divided by E; that is, the ratio of A to E.

Many have complained of this definition, and taxed Euclid with not only explaining one unknown thing by another, for he has not told us what the quantity of a ratio is, nor

nor what is to be understood by multiplying the quantities of the ratios into one another. But also for putting the definition as general to all quantities, when nevertheless it is only particular, *viz.* relates to the ratios of numbers and commensurable quantities.

But altho' Euclid cannot perhaps be intirely freed from blame in this matter, yet the difficulty of the subject, I mean the nature of ratios, requiring more to be said to be quite right in the matter, than he thought necessary to mention, and what he has mentioned, imperfect as it is, is so far intelligible and necessary, that his reader understands him to mean by compound ratio the multiplication of the quantities of simple ratios; and not their addition; and that he has said enough to make the following part of his work intelligible. I say, all this considered, will sufficiently excuse him. He knew it was difficult to give a better definition, and some one or other was to be given. The sixth book could not be understood without it; therefore he gave the best he could, consistent with brevity, and the intelligence of the following part of his work.

Scarborough, in several passages of his observations upon this fifth definition, misrepresents the affair. He says, amongst other things, 1. that the tenth definition of the fifth book is a sufficient definition of compound ratio. And secondly, that this fifth definition is never used in these *Elements* of Euclid, or in the *Conics* of Apollonius, or elsewhere in Archimedes. But in all these and other geometrical writers, composition of proportion is ever taken in the sense and notion of def. 10. lib. v. and to make this false assertion appear in some measure good, Scarborough, in his demonstration of the twenty third proposition of this sixth book, makes Euclid to say (see the figure) *But the ratio of K to M is compounded of the ratio of K to L, and of L to M, according to def. 10. of the fifth book*; when at the same time Euclid himself really says it is so by the fifth definition of the sixth book. Wherefore Euclid himself plainly uses this definition in the twenty third proposition of the sixth book. So does Apollonius, and other geometrical writers. And as to the tenth definition of the fifth book, it is only a particular case of compound ratio, *viz.* where the ratios are equal, and so is useless where the ratios are unequal; whereas this fifth definition extends to any unequal ratios whatsoever.

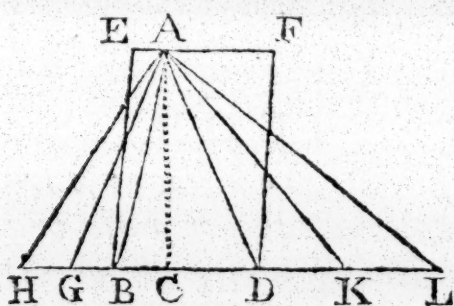
PROPOSITION I. THEOREM.

Triangles and parallelograms which have the same altitude, have the same ratio to one another, as their bases^d.

Let the triangles ABC , ACD , and the parallelograms EC , CF have the same altitude, *viz.* the perpendicular drawn from the point A to BD : I say as the base BC is to the base CD , so is the triangle ABC to the triangle ACD , and the parallelogram EC to the parallelogram CF .

For continue out BD both ways to the points H , L , and take any number of right lines BG , GH each equal to the base BC , and any number DK , KL each equal to the base CD . And join AG , AH , AK , AL .

Then because CB , BG , GH , are equal to one another; [by 38. I.] the triangles AGH , AGB , ABC will be equal to one another: Therefore the base HC is the same multiple of the base BC , that the triangle AHC is of the triangle ABC : By the same



reason the base LC is the same multiple of the base CD , that the triangle ALC is of the triangle ACD ; and [by 38. I.] if the base HC be equal to the base CL , the triangle AHC is equal to the triangle ALC :

If the base HC exceeds the base CL , the triangle AHC will also exceed the triangle ALC ; and if the base HC be less than the base CL ; the triangle AHC will likewise be less than the triangle ALC . There are therefore four magnitudes, *viz.* the two bases BC , CD , and the two triangles ABC , ACD , and there are equimultiples taken of the base BC , and the triangle ABC , *viz.* HC of the base, and the triangle AHC : as also other equimultiples of the base CD , and triangle ACD , *viz.* CL of the base and the triangle ALC . And it has been proved if the base HC exceeds the base CL , the triangle AHC exceeds the triangle ALC ; and if the base HC be equal to the base CL , the triangle AHC is equal to the triangle ALC : If the one be less, the other will be less too. Therefore [by 5. def. 5.] as the

base

base BC is to the base CD , so is the triangle ABC to the triangle ACD .

And because [by 41. 1.] the parallelogram EC is double to the triangle ABC , and the parallelogram CF is double to the triangle ACD . But [by 15. 5.] parts have the same ratio as their equimultiples; the triangle ABC will be to the triangle ACD , as the parallelogram EC is to the parallelogram FC . Therefore because it has been proved, as the base BC is to the base CD , so is the triangle ABC to the triangle ACD , but as the triangle ABC is to the triangle ACD , so is the parallelogram EC to the parallelogram FC : Therefore [by 11. 5.] as the base BC is to the base CD , so will the parallelogram EC be to the parallelogram FC .

Therefore triangles and parallelograms, which have the same altitude, will have the same ratio to one another as their bases. Which was to be demonstrated.

^d The demonstration of this proposition is one very natural and easy consequence or case of Euclid's fifth definition of his fifth book in triangles and parallelograms of the same altitude; shewing the exceeding easy application and usefulness of that definition.

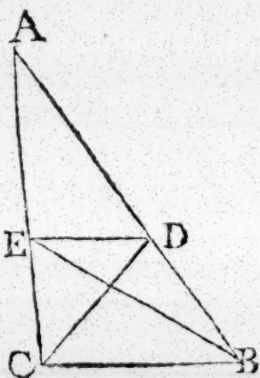
PROP. II. THEOR.

If a right line be drawn parallel to one side of a triangle, this will cut the sides of the triangle proportionally: and if the sides of a triangle be cut proportionally, the right line which joins the points of section will be parallel to the uncut side of the triangle.

For let the right line DE be drawn parallel to one side BC of a triangle ABC : I say as BD is to DA , so is CE to EA .

For join BE , CD .

Then [by 37. 1.] the triangle BDE is equal to the triangle CDE ; for they stand upon the same base DE , and are between the same parallels DE , BC . But ADE is another triangle; and [by 7. 5.] equal magnitudes have the same



ratio to the same magnitude: Therefore as the triangle BDE is to the triangle ADE , so is the triangle CDE , to the triangle ADE . But as the triangle BDE is to the triangle ADE , so is BD to DA : For since they have the same altitude, viz. the perpendicular drawn from the point E to AB , they are [by I. 6.] to one another, as their bases. And for the same reason the triangle CDE is to the triangle ADE , as CE is to EA : Therefore as BD is to DA , so is [by II. 5.] CE to EA .

Again, let the sides AB , AC of the triangle ABC be cut proportionally in the points D , E , so that BD be to DA , as CE is to EA : and join DE : I say DE is parallel to BC .

For the same construction remaining, because as BD is to DA , so is CE to EA : But as BD is to DA , so [by I. 6.] is the triangle BDE to the triangle ADE ; and as CE is to EA , so is the triangle CDE to the triangle ADE : it will be [by II. 5.] as the triangle BDE is to the triangle ADE , so is the triangle CDE to the triangle ADE . Therefore the triangles BDE , CDE have each the same ratio to the triangle ADE : Wherefore [by 9. 5.] the triangle BDE is equal to the triangle CDE . But they stand both upon the same base DE . And equal triangles that stand upon the same base [by 39. I.] are also between the same parallels. Wherefore DE is parallel to BC .

If therefore a right line be drawn parallel to one side of a triangle, cutting the other two sides; this line will cut the sides proportionally: and if the sides of a triangle be cut proportionally, a right line joining the points of section, will be parallel to the uncut side of the triangle. Which was to be demonstrated.

PROP. III. THEOR.

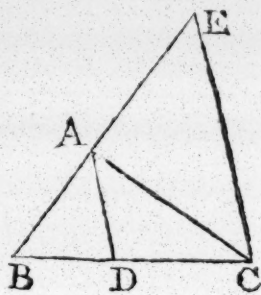
If one angle of a triangle be bisected, and the bisecting right line cuts the base, the segments of the base will have the same ratio to one another that the other two sides of the triangle have to one another: and if the segments of the base have the same ratio to one another, as the other two sides of the triangle have to one another: then will the right line drawn from the point of section to the

vertex of the triangle bisect the angle at the vertex.

Let there be a triangle ABC , and let the angle BAC be cut into two equal parts by the right line AD : I say, as BD is to DC , so is BA to AC .

For [by 31. 1.] draw CE thro' C parallel to DA , and let BA continued out meet the same in E .

Then because the right line AC falls upon the two parallels AD, EC ; the angle ACE [by 29. 1.] will be equal to the angle CAD . But the angle CAD is supposed to be equal to the angle BAD : Therefore BAD will also be equal to the angle ACE . Again, because the right line BAE falls upon the parallel right lines AD, EC , the outward angle BAD [by 29. 1.] is equal to the inward angle AEC . And it has been also proved that the angle ACE is equal to the angle BAD : Therefore ACE will also be equal to the angle AEC : and so [by 6. 1.] the side AE is equal to the side AC . And because AD is drawn parallel to one side, *viz.* EC of the triangle BCE ; [by 2. 6.] it will be as BD is to DC , so is BA to AE . But AE is equal to AC : Therefore [by 7. 5.] as BD is to DC , so is BA to AC .



But now let BD be to DC , as BA is to AC : and join AD : I say the angle BAC is bisected by the right line AD .

For the same construction remaining, because it is as BD to DC , so is BA to AC ; and [by 2. 6.] as BD is to DC , so is BA to AE (for AD is drawn parallel to one side EC of the triangle BCE) it will be as BA to AC , so is BA to AE : Therefore [by 9. 5.] AC is equal to AE , and so [by 5. 1.] the angle AEC is equal to the angle ACE . But the angle AEC [by 29. 1.] is equal to the outward angle BAD , and the angle ACE equal to the alternate angle CAD : Therefore the angle BAD will be equal to the angle CAD : Wherefore the angle BAC is bisected by the right line AD .

Therefore if one angle of a triangle be bisected, and the bisecting line also cuts the base; the segments of the base will have the same ratio, as the other two sides of the triangle. And if the segments of the base have the same

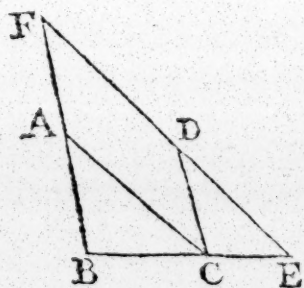
ratio as the other sides of the triangle, the right line drawn from the vertex to the point of section, will bisect the angle at the vertex of the triangle. Which was to be demonstrated.

PROP. IV. THEOR.

The sides about the equal angles of equiangular triangles, are proportional: and the sides opposite to the equal angles are homologous or co-rationals.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE , and the angle ACB equal to the angle DEC , and moreover the angle BAC equal to the angle CDE : I say the sides about the equal angles of the triangles ABC, DCE are proportional; and the sides opposite to the equal angles are homologous or co-rationals.

For let BC, CE be both put in the same right line: and because [by 17. 1.] the angles ABC, ACB are less than two right angles. But the angle ACB is equal to DEC ; the angles ABC, DEC will be less than two right angles: Wherefore [by 11. ax.] BA, ED being produced will meet one another. Let them be produced and meet in F .



Then because the angle DCE is equal to the angle ABC ; [by 28. 1.] BF will be parallel to CD . Again, because the angle ACB is equal to the angle DEC ; AC will be parallel to FE : Therefore $FACD$ is a parallelogram: And so [by 34. 1.] FA is equal to CD , and AC to FD . And because AC is drawn parallel to one

side FE of the triangle FBE ; [by 2. 6.] it will be as BA is to AF , so is BC to CE . But AF is equal to CD : Therefore as BA is to CD , so is [by 7. 5.] BC to CE : and alternately [by 16. 5.] as AB is to BC , so is DC to CE . Again, because CD is parallel to BF , it will be as BC is to CE , so is FD to DE . But DF is equal to AC : Therefore as BC is to CE , so is AC to ED . And alternately as BC is to CA , so is CE to ED . And so because it has been proved that as AB is to BC , so is DC to CE , and

as BC is to CA , so is CE to ED ; it shall be by equality [by 22. 5.] as BA is to AC , so is CD to DE .

Therefore the sides about the equal angles of equiangular triangles, are proportional; and the sides opposite to the equal angles are homologous or co-rationals. Which was to be demonstrated.

PROP. V. THEOR.

If two triangles have their sides proportional; those triangles will be equiangular; those angles being equal which are opposite to the homologous or co-rational sides.

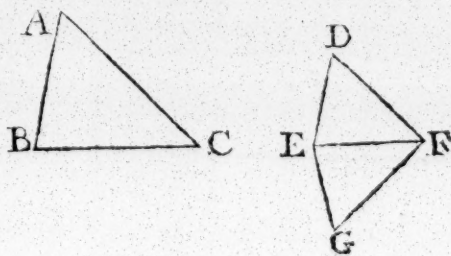
Let there be two triangles ABC , DEF , having proportional sides: and let AB be to BC , as DE is to EF : also as BC is to CA , so is EF to FD , and moreover as BA is to AC , so is ED to DF : I say the triangles ABC , DEF are equiangular, those angles being equal which are opposite to the homologous sides, *viz.* the angle ABC equal to the angle DEF , the angle BCA equal to the angle EFD , and the angle BAC equal to the angle EDF .

For [by 23. 1.] at the right line EF , and at the points E , F in it, make the angle FEG equal to the angle ABC , and the angle EFG , equal to the angle BCA ; then [by 32. 1.] the remaining angle BAC is equal to the remaining angle EGF .

Therefore the triangles ABC , EGF are equiangular; and so the sides about the equal angles of the triangles ABC , EGF are [by 4. 6.] proportional: and the sides opposite to the equal angles are homologous: Wherefore as

AB is to BC , so is GE to EF . But as AB is to BC , so is DE to EF : Therefore as DE is to EF , so [by 11. 5.] is GE to EF : Wherefore DE , GE have each the same ratio to EF :

And so [by 9. 5.] DE will be equal to GE . By the same reason DF will also be equal to GF . Then because DE is equal to EG , and EF is common, the two sides DE , EF , are equal to the sides GE , EF , and the base DF is equal to the base GF : Therefore [by 8. 1.] the angle DEF is equal



equal to the angle $G E F$, and the triangle $D E F$ equal to the triangle $G E F$, and the remaining angles of the one triangle equal to the remaining angles of the other, which are opposite to the equal sides. Therefore the angle $D F E$ is equal to the angle $G F E$, and the angle $E D F$ equal to the angle $E G F$. And because the angle $D E F$ is equal to the angle $G E F$, and [by construction] the angle $G E F$ is equal to the angle $A B C$; the angle $A B C$ will be equal to the angle $D E F$. By the same reason the angle $A C B$ is equal to the angle $D F E$, and also the angle A equal to the angle D : Therefore the triangles $A B C$, $D E F$ are equiangular.

If therefore two triangles have their sides proportional, the triangles will be equiangular: and those angles will be equal which are opposite to the homologous sides. Which was to be demonstrated.

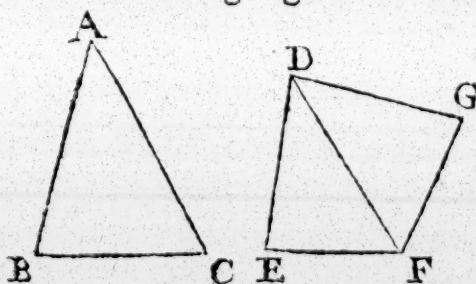
PROP. VI. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportional: these triangles will be equiangular, having those angles equal which are opposite to the homologous sides.

Let two triangles $A B C$, $D E F$ have one angle $B A C$ of the one, equal to one angle $E D F$ of the other, and the sides about these equal angles proportional, viz. as $B A$ is to $A C$, so is $E D$ to $D F$: I say the triangles $A B C$, $D E F$ are equiangular, the angle $A B C$ being equal to the angle $D E F$, and the angle $A C B$ equal to the angle $D F E$.

For [by 23. 1.] at the right line $D F$, and at the points D , F in it, make the angle $F D G$ equal to $B A C$ or $E D F$, and the angle $D F G$ equal to the angle $A C B$.

Then will the remaining angle at B [by 32. 1.] be equal to the remaining angle at G . Therefore the triangles $A B C$, $D G F$ are equiangular: and so [by 4. 6.] as $B A$ is to $A C$, so is $G D$ to $D F$. But [by supposition] as $B A$ is to $A C$, so is $E D$ to $D F$: Therefore as $E D$ is to $D F$, so is [by 11. 5.]



GD to DF : Wherefore [by 9. 5.] ED is equal to DG , and DF is common: Therefore the two sides ED , DF are equal to the two sides GD , DF , and the angle EDF is equal to the angle GDF : Wherefore the base EF [by 4. 1.] is equal to the base FG , and the triangle DEF equal to the triangle GDF , and the remaining angles equal to the remaining angles, each to each, which are opposite to the equal sides: Therefore the angle DFG is equal to the angle DFE ; and the angle at G equal to the angle at E . But the angle DFG [by construction] is equal to the angle ACB : Therefore the angle ACB is equal to the angle DFE . But BAC is supposed to be equal to the angle EDF : Therefore [by 32. 1.] the remaining angle at B is equal to the remaining angle at E : Wherefore the triangles ABC , DEF are equiangular.

Therefore if two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportional; these triangles will be equiangular, having those angles equal which are opposite to the homologous sides. Which was to be demonstrated.

PROP. VII. THEOR.

If two triangles have one angle of the one equal to one angle of the other, and the sides about other angles proportional, and have also each of the remaining angles either less or not less than a right angle: those triangles will be equiangular, and have those angles equal about which are the proportional sides^e.

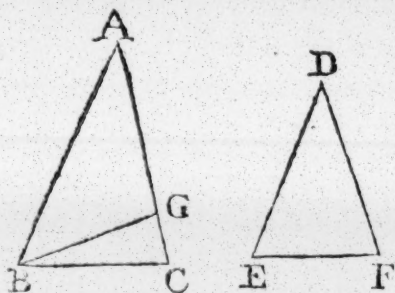
Let the two triangles be ABC , DEF , having one angle BAC of the one, equal to one angle EDF of the other, and the sides about other angles ABC , DEF proportional, so that DE be to EF , as AB is to BC . And having also each of the remaining angles at C , F first less than a right angle: I say the triangles ABC , DEF will be equiangular, the angle ABC equal to the angle DEF , and the remaining angle at C equal to the remaining angle at F .

For if the angle ABC be unequal to the angle DEF , one of them will be the greater, which let be ABC . And

at

at the right line AB and at the point B in it make [by 23. 1.] the angle ABG equal to the angle DEF .

Now because the angle A is equal to the angle D , and the angle ABG equal to the angle DEF ; [by 32. 1.] the remaining angle AGB will be equal to the remaining angle DFE : Therefore the triangle ABG is equiangular to the triangle DEF : Wherefore [by 4. 6.] as AB is to BG , so is DE to EF . But as DE is to EF , so is [by supposition] AB to BC : Therefore as AB



is to BC , so [by 11. 5.] is AB to BG ; and so AB has the same ratio to BC or BG : Wherefore BC is equal to BG ; and accordingly [by 5. 1.] the angle BGC is equal to the angle BGG . But the angle at C is supposed to be less than a right angle: Therefore BGC will also be less than a right angle, wherefore [by 13. 1.] the adjacent angle AGB will be greater than a right angle. But it has been proved that the angle AGB is equal to the angle at F : Therefore the angle at F is greater than a right angle. But it is supposed to be less than a right angle; which is absurd: Therefore the angle ABC is not unequal to the angle DEF ; they are therefore equal. But the angle at A is equal to the angle at D ; wherefore the remaining angle at C is equal to the remaining angle at F : Therefore the triangles ABC , DEF are equiangular.

Again, let us suppose each of the angles C , F to be not less than a right angle: I say moreover that the triangles ABC , DEF will be equiangular.

For the same construction remaining, we demonstrate after the like manner that BC is equal to BG , and the angle C equal to the angle BGC . But the angle at C is not less than a right angle; Therefore BGC is not less than a right angle: Wherefore the two angles of the triangle BGC are not less than two right angles, which [by 17. 1.] is impossible: Wherefore again the angle ABC is not unequal to the angle DEF : it is therefore equal to it. But the angle at A is equal to the angle at D : Therefore the remaining angle at C [by 32. 1.] is equal to the remaining angle at F : and accordingly the triangles ABC , DEF are equiangular.

If therefore two triangles have one angle of the one equal to one angle of the other, and the sides about other angles proportional, and have also each of the remaining angles either less or greater than a right angle: these triangles will be equiangular, and shall have those angles equal, about which the sides are proportional. Which was to be demonstrated.

* Euclid has added that each of the remaining angles B, E must be either both less or both not less than a right angle, for otherwise, the whole hypothesis remaining, it would not follow that the triangles would be equiangular. For if in the same triangle ABC , BC were equal to AC , which would happen when the angle B is acute, and greater than A : In this case the two triangles ABC, DEF would have one angle of the one equal to one angle of the other, *viz.* the angle A and the angle D , and the sides about the other angles ABC, DEF proportional, that is, as AB is to BC , so is DE to EF , and yet the triangles ABC, DEF would not be equiangular; and that because one angle ACB is greater than a right angle, and the other EFD less.

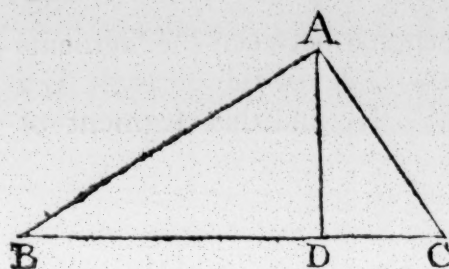
Taquet says this proposition is scarcely of any use. But herein he is more prompted by dislike than certainty.

PROP. VIII. THEOR.

If a perpendicular be drawn from the right angle to the base of a right angled triangle: this perpendicular will divide that right angled triangle into two other triangles; each of which will be similar to one another, and to the whole right angled triangle^f.

Let there be a right angled triangle ABC , having the right angle BAC ; and let AD be the perpendicular drawn from the point A to the base BC : I say the triangles ABD, ADC are similar to one another, and also to the whole triangle ABC .

For because the angle BAC is equal to the angle ADB , each of them being a right angle, and the angle which is at E is common to both the triangles ABC, ABD ; the remaining angle ACB [by 32. 1.] will be equal to the remaining angle BAD : Therefore the triangle ABC is equiangular to the triangle ABD . Wherefore [by 4. 6.] as the



the side BC opposite to the right angle of the triangle ABC , is to the side BA , opposite to the right angle of the triangle ABD , so is the side AB opposite to the angle at C of the triangle ABC to the side BD opposite to an angle equal to the angle at C , viz. BAD of the triangle ABD . And so also is AC to the side AD opposite to the angle at B , which is common to both the triangles: Therefore the triangle ABC is equiangular to the triangle ABD , and has the sides about the equal angles proportional: therefore [by I. def. 6.] the triangle ABC is similar to the triangle ABD . By the same reason we demonstrate that the triangle ADC is also similar to the triangle ABC ; wherefore the triangles ABD , ADC are each similar to the whole triangle ABC .

I say moreover that the triangles ABD , ADC are also similar to one another.

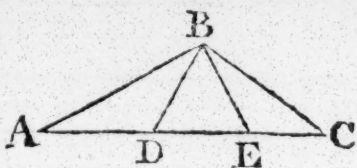
For because the right angle BDA is equal to the right angle ADC ; but BAD has been proved to be equal to the angle at C ; [by 32. 1.] the remaining angle at B will be equal to the remaining angle DAC : Therefore the triangle ABD is equiangular to the triangle ADC . Wherefore [by 4. 6.] as the side BD of the triangle ABD which is opposite to the angle BAD , is to the side DA of the triangle ADC , which is opposite to the angle at C , equal to the angle BAD , so is the side AD of the triangle ABD , which is opposite to the angle at B , to the side DC of the triangle ADC , which is opposite to the angle DAC , being equal to the angle at B . And so also is the side BA opposite to the right angle ADB , to the side AC which is opposite to the right angle ADC : Therefore [by I. def. 6.] the triangle ABD is similar to the triangle ADC .

Wherefore if a perpendicular be drawn from the right angle to the base of a right angled triangle; this perpendicular will divide that right angled triangle into two other triangles, each of which will be similar to one another, and to the whole right angled triangle. Which was to be demonstrated.

Corollary. From hence it is manifest, that the perpendicular drawn from the right angle to the base of a right angled

gled triangle, is a mean proportional between the segments of the base: and moreover either side of the triangle is a mean proportional between the base, and that segment of the base, which is next to it.

Euclid does not consider the following case, *viz.* in any triangle ABC , if the right lines BD, BE be so drawn from the angle B to the base AC , that the angles ADB, BEC be each equal to the angle ABC of the triangle ABC . The triangles ABD, BEC will be similar to one another, and to the whole triangle ABC . And the square of the side AB will be equal to the rectangle under the base AC of the whole triangle and the segment AD of it, and the square of the side BC equal to the rectangle under the whole base AC and the segment EC of it, and the sum of the squares of the sides AB, BC will be equal to the rectangle under the base AC , and the sum of the segments AD, EC of the base. The demonstration is very easy, from what Euclid has said in this proposition, and prop. 17. of this book.

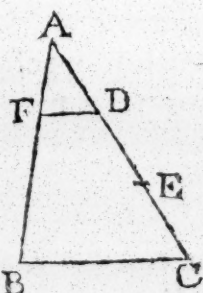


PROP. IX. PROBL.

To cut off a part required from a given right line.

Let the given right line be AB : to cut off from it a part required.

Let the part required be one third; draw any right line AC from the point A , making any angle with AB ; assume any point D in AC , and [by 3. 1.] make DE, EC each equal to AD . Join BC , and draw [by 31. 1.] DF thro' D parallel to BC .



Then because the right line FD is drawn parallel to one side BC of a triangle ABC ; [by 2. 6.] it will be as CD is to DA , so is BF to FA . But CD is double to DA : therefore BF is also double to FA : Consequently BA is thrice AF .

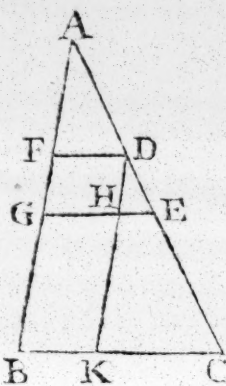
Wherefore a third part AF required is cut off from the given right line AB . Which was to be done.

PROP.

PROP. X. PROBL.

To cut a given [undivided] right line into parts that shall be similar to those of a given divided right line^s.

Let AB be a given undivided right line, and AC a given divided right line. It is required to cut the right line AB into parts similar to those that AC is divided into.



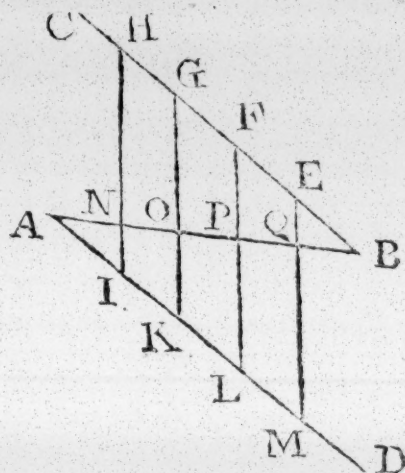
Let AC be divided in the points D, E , and so join AC, AB as to contain any angle. Join B, C ; and [by 31. I.] thro' the points D, E , draw DF, EG parallel to BC , and thro' D draw DH, HK parallel to AB .

Then FH, HB are each of them parallelograms: and accordingly [by 34. I.] DH is equal to FG , and HK to GB . And because HE is drawn parallel to one side KC of the triangle DKC ; it will be [by 2. 6.] as CE is to ED , so is KH to HD . But KH is equal to BG ; and HD to FG : Therefore as CE is to ED , so is BG to GF . Again, because FD is drawn parallel to one side EG of the triangle AGE : as ED is to DA , so will GF be to FA . But it has been proved that CE is to ED , as BG is to GF : Therefore as CE is to ED , so is BG to GF , and as ED is to DA , so is GF to FA .

Therefore the given undivided right line AB is cut into parts similar to the parts that the given right line AC is divided into. Which was to be done.

§ From this problem it is easy to find out and demonstrate the way of dividing a given right line, as AB into any proposed number of equal parts. For let the proposed number of equal parts be five. From the ends A and B of the given right line, draw the right lines AD, BC parallel to one another; that is, make the angles A, B equal to one another: and upon AD, BC each take at pleasure from A towards D , four equal parts, AI, IK, KL, LM , and from B towards C take four more of them BE, EF, FG, GH , the number of parts on each being less by one than the proposed number of equal parts. This done, join the points I, H ; K, G ; L, F ; M, E , by right lines

lines cutting the given right line AB in the points N, O, P, Q . Then will the proposed right line AB be divided into five equal parts AN, NO, OP, PQ, QB . For since GH, IK are equal and parallel; the right lines HI, CK will [by 33. 1.] also be equal and parallel. By the same reason GK, FL, EM will be parallel. Wherefore since AM is divided into four equal parts, AQ will also be



as is manifest from the demonstration of this tenth proposition. By the same reason BN will be divided into four equal parts, because BH is divided into the like number of equal parts. Wherefore since both AN, BQ are each equal to NO, OP, PQ ; all the five parts AN, NO, OP, PQ, QB will be equal to one another. Which was to be done.

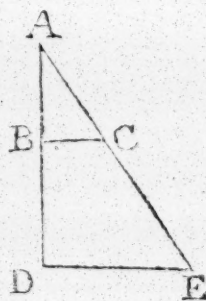
There are several ways to perform this problem. But none so elegant, as this way.

PROP. XI. PROBL.

To find a third right line proportional to two given right lines.

Let the two given right lines be AB, AC , so placed as to make any angle. It is required to find a third right line proportional to AB, AC .

Produce AB, AC to the points D, E , so that BD be equal to AC . Join BC , and [by 31. 1.] draw DE thro' D parallel to BC .



Then because BC is drawn parallel to one side DE of the triangle ADE ; [by 2. 6.] it will be as AB is to BD , so is AC to CE . But BD is equal to AC : Therefore as AB is to AC , so is AC to CE .

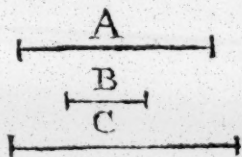
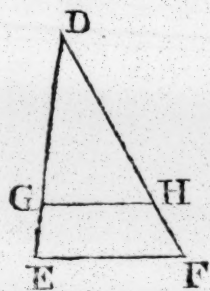
Wherefore there is found a third proportional CE to the two given right lines AB, AC . Which was to be done.

PROP.

PROP. XII. THEOR.

Three right lines being given : to find a fourth right line proportional to them.

Let the three given right lines be A, B, C . It is required to find a fourth right line proportional to A, B, C .



Draw two right lines DE, DF containing any angle EDF , and make DE equal to A , GE equal to B , and DH equal to C : Join GH , and thro' E draw EF parallel to it.

Then because GH is drawn parallel to one side EF of the triangle DEF ; [by 2. 6.] it will be as DE is to GE , so is DH to HF . But DE is equal to A , GE equal to B , and DH equal to C : Therefore as A is to B , so is C to HF .

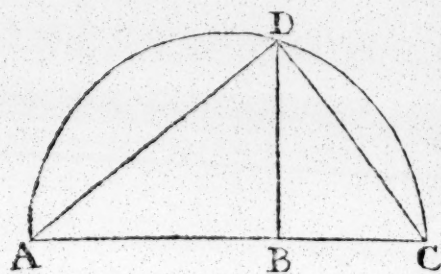
Wherefore there is found a right line HF which is a fourth proportional to three given right lines A, B, C . Which was to be done.

PROP. XIII. THEOR.

Two right lines being given : to find a mean proportional between them.

Let the two given right lines be AB, BC . It is required to find a mean proportional line between them.

Place them both in the same direction, and describe the



femicircle ADC upon the right line AC , and [by 11.1.] from the point B draw BD at right angles to AC , and join AD, DC .

Then because the angle ADC is in a semicircle, [by 31. 3.] it is a right angle. And because DB is drawn from the right angle of a right angled triangle ADC perpendicular to the base [by cor. 8. 6.] DB will be a mean proportional between the segments AB, BC of the base.

Therefore there is found a mean proportional DB between two given right lines AB, BC . Which was to be done.

PROP.

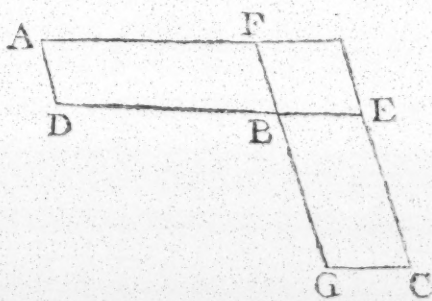
PROP. XIV. THEOR.

If equal parallelograms have one angle of the one equal to one angle of the other: the sides about the equal angles will be reciprocally proportional: and if parallelograms have one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional: those parallelograms are equal to one another.

Let the equal parallelograms be AB, BC having the angles at B equal. And let DB, BE be both placed in the same right line: Then [by 14. 1.] FB, BG will be both in the same right line: I say the sides of the parallelograms AB, BC , which are about the equal angles are reciprocally proportional: that is, as DB is to BE , so is GB to BF .

For completat the parallelogram FE .

Then because the parallelogram AB is equal to the parallelogram BC , and FE is some other parallelogram: [by 7. 5.] it will be as AB is to FE , so is BC to FE . But [by 1. 6.] as AB is to FE , so is DB to BE , and as BC is to FE , so is GB to BF : Therefore [by 11. 5.] it will be as DB is to BE , so is GB to BF . Therefore the sides of the parallelograms AB, BC which are about the equal angles, are reciprocally proportional.



But now let the sides, about the equal angles, be reciprocally proportional, viz. as DB is to BE , so is GB to BF : I say the parallelogram AB is equal to the parallelogram BC .

For because it is as DB is to BE , so is GB to BF and as DB is to BE , so [by 1. 6.] is the parallelogram AB to the parallelogram FE , and as GB is to BF , so is the parallelogram BC to the parallelogram FE : [by 11. 5.] it will be as AB is to FE , so is BC to FE . Therefore [by 9. 5.] the parallelogram AB is equal to the parallelogram BC .

§

Therefore

Therefore if equal parallelograms have one angle of the one equal to one angle of the other; the sides about the equal angles will be reciprocally proportional. And if parallelograms have one angle of the one equal to one angle of the other, and the sides about these equal angles reciprocally proportional; these parallelograms are equal to one another. Which was to be demonstrated.

PROP. XV. THEOR.

If equal triangles have one angle of the one equal to one angle of the other; the sides about the equal angles will be reciprocally proportional: and if triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional, those triangles are equal to one another.

Let the equal triangles be ABC , ADE , having one angle BAC of the one equal to one angle DAE of the other. I say the sides of the triangles ABC , ADE which are about the equal angles are reciprocally proportional, that is, as CA is to AD , so is EA to AB .

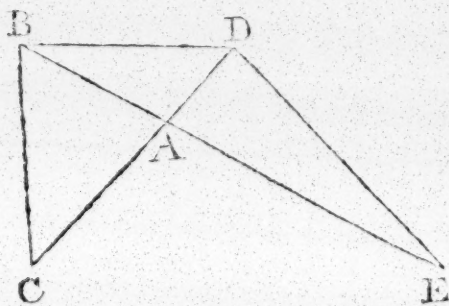
For so place them that CA , AD both lie in the same right line: Therefore [by I. 4. 1.] EA , AB will be also both in one right line. Join BD .

Then because the triangle ABC is equal to the triangle ADE , and ABD is some other triangle; it will be [by 7. 5.] as the triangle CAB is to the triangle BAD , so is the triangle ADE to the triangle BAD . But [by 1. 6.]

as the triangle CAB is to the triangle BAD , so is CA to AD . And as the triangle EAD is to the triangle BAD , so is EA to AB : Therefore [by 11. 5.] as CA is to AD , so is EA to AB : Wherefore the sides of the triangles ABC , ADE ,

which are about the equal angles, are reciprocally proportional.

Now



Now let the sides of the triangles ABC , ADE be reciprocally proportional; and let it be as CA is to AD , so is EA to AB : I say the triangle ABC is equal to the triangle ADE .

For again join BD . Then because as CA is to AD , so is EA to AB ; and as CA is to AD , so [by I. 6.] is the triangle ABC to the triangle BAD , and as EA is to AB , so is the triangle EAD to the triangle BAD : [by II. 5.] it will be as the triangle ABC is to the triangle BAD , so is the triangle EAD to the triangle BAD : Therefore the triangles ABC , ADE have each the same ratio to the triangle BAD : and accordingly [by 9. 5.] the triangle ABC is equal to the triangle EAD .

Therefore if equal triangles have one angle of the one equal to one angle of the other; the sides about the equal angles will be reciprocally proportional: and if triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional; these triangles are equal to one another. Which was to be demonstrated.

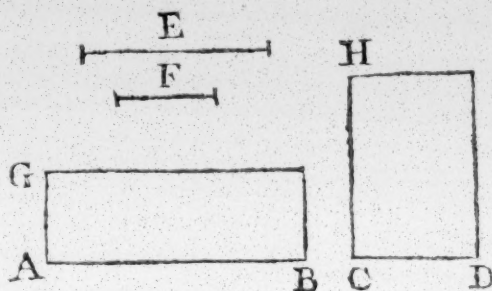
PROP. XVI. THEOR.

If four right lines be proportional, the rectangle contained under the extremes is equal to the rectangle contained under the means: and if the rectangle contained under the extremes be equal to the rectangle contained under the means, the four right lines will be proportional.*

Let the four right lines AB , CD , E , F be proportional, viz. as AB is to CD , so is E to F : I say the rectangle contained under the right lines AB , F , is equal to the rectangle contained under CD , E .

For [by II. 1.] from the points A , C draw AG , CH at

* It should be added, *the first to the second, and the third to the fourth*; otherwise the proposition may not be true. For these four numbers 2, 6, 4, 3 are proportionals, altho' they be misplaced, and the product of the extremes 2 and 3 is not equal to the product of the means 6 and 4. The same caution should be observed in the twelfth proposition, which perhaps might be better express'd thus: *Three right lines being given: to find a fourth such, that the ratio of any two of them shall be equal to the ratio of the two remaining right lines.*



right angles to AB, CD , and make AG equal to F , and CH equal to E ; and compleat the parallelogram BG, DH .

Then because it is as AB is to CD , so is E to F , and E is equal to CH , and F to AG ; it will be [by 7. 5.] as AB is to CD , so is CH to AG : Therefore the sides about the equal angles of the parallelograms BG, DH , are reciprocally proportional. But if parallelograms have the sides about the equal angles reciprocally proportional, [by 14. 6.] these parallelograms are equal to one another: Therefore the parallelogram BG is equal to the parallelogram DH . But the parallelogram BG is the rectangle contained under the right lines AB, F , for AG is equal to F ; and the parallelogram DH is the rectangle contained under the right lines CD, E , since CH is equal to E . Therefore the rectangle contained under the right lines AB, F is equal to the rectangle contained under the right lines CD, E .

But now let the rectangle contained under the right lines AB, F be equal to the rectangle contained under the right lines CD, E : I say the four right lines are proportional; that is, as AB is to CD , so is E to F .

For the same construction remaining; because the rectangle contained under the right lines AB, F is equal to the rectangle contained under the right lines CD, E , and the rectangle BG is that contained under the right lines AB, F , for AG is equal to F : but the rectangle contained under the right lines CD, E , is the rectangle DH , since CH is equal to E : the parallelogram BG will be equal to the parallelogram DH ; they are also equiangular. But the sides of equal parallelograms that are about the equal angles, are [by 14. 6.] reciprocally proportional: Wherefore as AB is to CD , so is CH to AG . But CH is equal to E , and AG to F : Therefore as AB is to CD , so is E to F .

Therefore if four right lines be proportional, the rectangle contained under the extremes, is equal to the rectangle contained under the means: and if the rectangle contained under the extremes be equal to the rectangle contained under the means; the four right lines will be proportional.

PROP.

PROP. XVII. THEOR.

If three right lines be proportional: the rectangle contained under the extremes is equal to the square of the mean line: and if the rectangle contained under the extremes be equal to the square of the mean line, the three right lines will be proportional.

Let the three right lines A, B, C be proportional; so that A be to B, as B is to C: I say the rectangle contained under the right lines A, C is equal to the square of the mean line B.

For let D be put equal to B.

Then because as A is to B, so is B to C, and B is equal to D; it will be as A is to B, so is D to C. But if four

right lines be proportional, the rectangle contained under the extremes [by 16. 6.]

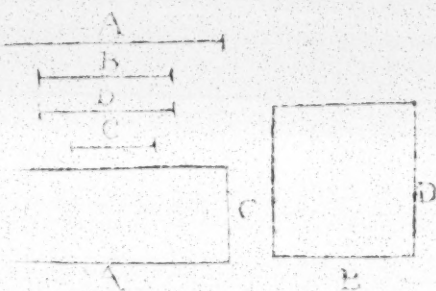
is equal to the rectangle contained under the means:

Therefore the rectangle contained under the right lines A, C is equal to the

rectangle contained under the right lines B, D. But the rectangle contained under the right lines B, D is equal to the square of B; for B is equal to D: Therefore the rectangle contained under the right lines A, C is equal to the square of B.

But now let the rectangle contained under the right lines A, C be equal to the square of B: I say it will be as A is to B, so is B to C.

For the same construction remaining, because the rectangle contained under the right lines A, C, is equal to the square of B; but the square of B is the rectangle contained under B, D; for B is equal to D, the rectangle contained under the right lines A, C is equal to the rectangle contained under the right lines B, D. But if the rectangle contained under the extremes be equal to the rectangle contained under the means; [by 16. 6.] the four right lines are proportional: Therefore as A is to B, so is D



to c : But b is equal to d : Wherefore as A is to B , so is B to C .

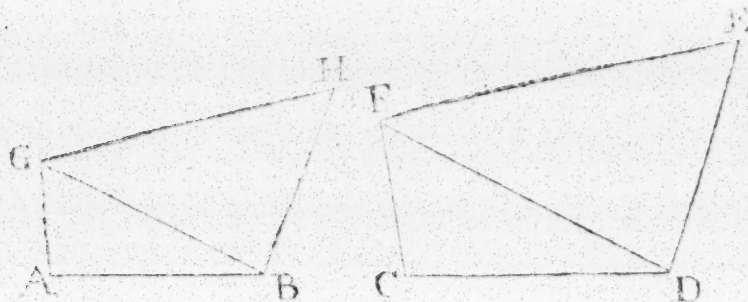
If therefore three right lines be proportional, the rectangle contained under the extremes will be equal to the square of the mean line: and if the rectangle contained under the extremes be equal to the square of the mean line, the three right lines will be proportional. Which was to be demonstrated.

PROP. XVIII. THEOR.

Upon a given right line, to describe a right-lined figure similar and alike situate to a given right-lined figure^h.

Let AB be the given right line, and $CDEF$ the given right-lined figure: it is required to describe a right-lined figure upon AB similar and alike situate to the given right-lined figure $CDEF$.

Join DF ; and [by 23. 1.] at the right line AB , and at the points A, B in it, make the angle GAB equal to the angle C , and the angle ABG equal to the angle CDF : Then [by 32. 1.] the angle remaining CFD is equal to the angle remaining AGB : Therefore the triangle FGD is equiangular to the triangle GAB ; and accordingly [by 4. 6.] as FD is to GB , so is FE to GA and CD to AB . Again, at the right line BG , and at the points B, G in it,



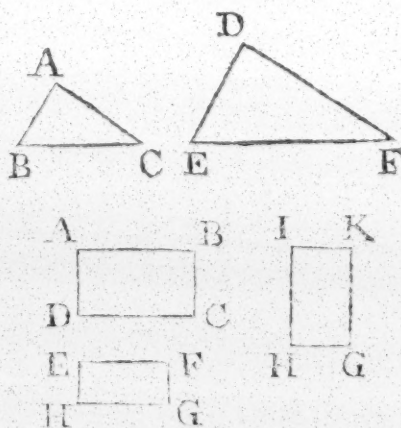
make the angle BGH equal to the angle DCE , and the angle GHB equal to the angle EDC : Then the remaining angle at E is equal to the remaining angle at H : Therefore the triangle FDE is equiangular to the triangle GHB : and so [by 4. 6.] as FD is to GB , so is FE to GH and ED to HB . But it has been also proved, that as

FD

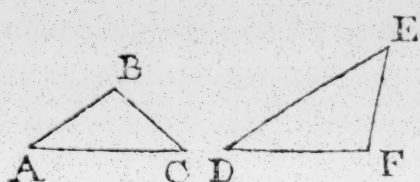
FD is to GB, so is FC to GA and CD to AB: and therefore [by II. 5.] as FC is to GA, so is CD to AB and FE to GH, and moreover ED to HB. Wherefore because the angle CFD [by construction] is equal to the angle AGB, and the angle DFE to the angle BGH: the whole angle CFE is equal to the whole angle AGH. By the same reason the angle CDE is also equal to the angle ABH, and besides the angle at C is equal to the angle at A; but the angle at E is equal to the angle at H; therefore AH is equiangular to CE, and it has the sides about the equal angles proportional: Wherefore [by I. def. 6.] the right lined figure AH will be similar to the right lined figure CE.

Therefore upon the given right line AB the right lined figure AH is described similar to the given right lined figure CE, and alike situate. Which was to be done.

^b Similar right-lined figures are said to be alike described, or set upon right lines, when those right lines are the homologous sides of the similar right lined figures; or when the equal angles are constituted upon those right lines, and the remaining equal angles, and proportional sides of the figures always follow one another in order. As the triangles ABC, DEF are not only similar but are alike situate upon the right lines BC, EF, when the angles B, C are equal to the angles E, F; and it be as AB to BC, so is DE to EF, &c. But they would not have been similarly situate, if the angle B had been equal to F, and C equal to E. In like manner the rectangles AC, EG are similar and alike situate upon the right lines DC, HG, when AD is to DC as EH is to HG &c. But the rectangles AC, IG are said not to be similar situate upon the right lines DC, HG altho' they are similar, as is manifest. But notwithstanding the same figures will be alike situate or described upon the right lines DC, IH, or upon AD, HG.



That the figures be alike situate, is a necessary condition. For if they be not, this proposition, and all the others following, limited to that condition, will not be always true, because two similar figures may be described upon the same right line not



alike situate, that will be unequal, as the triangles ABC , DEF , (where AC is equal to DF , and the angles $A, D; B, E$ are respectively equal to one another) will be unequal.

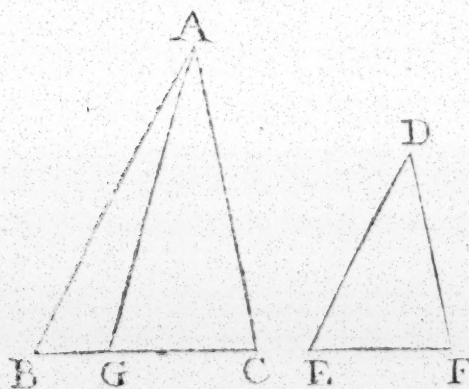
Note also that all equilateral and equiangular figures are alike described upon any right lines.

PROP. XIX. THEOR.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let the triangles ABC , DEF be similar, having the angle B equal to the angle E ; and let AB be to BC , as DE is to EF : so that [by 12. def. 5.] the side BC is homologous to the side EF : I say the triangle ABC is to the triangle DEF in the duplicate ratio of that of the side BC to the side EF .

For [by 11. 6.] find a third proportional BG to BC , EF , so that BC be to EF , as EF is to BG ; and join GA .



Then because it is as AB to BC , so is DE to EF ; it will be alternately [by 16. 5.] as AB is to DE , so is BC to EF . But as BC is to EF , so is EF to BG ; and therefore [by 11. 5.] as AB is to DE , so is EF to BG : Therefore the sides about the equal angles of the triangles

ABG , DEF , are reciprocally proportional. But those triangles having one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional, are [by 15. 6.] equal to one another: therefore the triangle ABG is equal to the triangle DEF . And because BC is to EF , as EF is to BG ; and if three right lines be proportional, the ratio of the first to the third is [by 10. def. 5.] duplicate to the ratio that the first has to the second: the ratio of BC to BG will be duplicate of the ratio of BC to EF . But as BC is to BG , so [by 1. 6.] is the triangle ABC to the triangle ABG : Therefore also the

the ratio of the triangle ABC to the triangle ABG is duplicate of the ratio of BC to EF . But the triangle ABG is equal to the triangle DEF : and therefore [by 7. 5.] the ratio of the triangle ABC to the triangle DEF is the duplicate of the ratio of BC to EF .

Therefore similar triangles are to one another in the duplicate ratio of their homologous sides. Which was to be demonstrated.

Corollary. From hence it is manifest, if three right lines be proportional, as the first is to the third, so is a triangle described upon the first, to a triangle similar and alike situated, described upon the second: because it has been proved as CB is to BG , so is the triangle ABC to the triangle ABG , that is, to the triangle DEF .

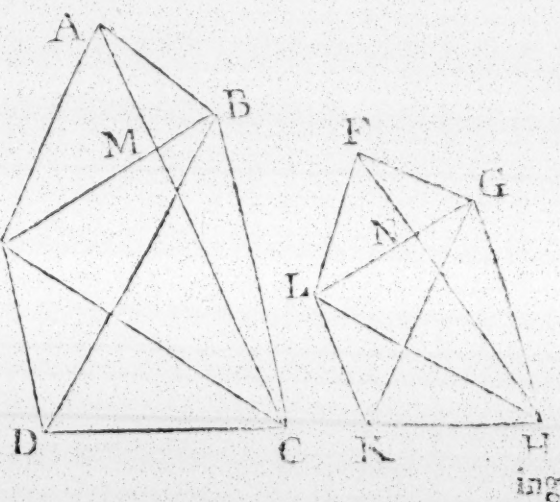
PROP. XX. THEOR.

Similar polygons are divided into equal numbers of similar triangles homologous to the wholes; and one polygon has to another polygon a duplicate ratio to that which one homologous side of the one has to an homologous side of the other.

Let the polygons $ABCDE$, $FGHKL$ be similar, and let AB , FG be homologous sides: I say the polygons $ABCDE$, $FGHKL$ are divided into an equal number of similar triangles, homologous to the wholes: and the polygon $ABCDE$ to the polygon $FGHKL$ has a duplicate ratio to that which one homologous side AB has to another FG .

Join BE , EC , GL , LH .

Then because the polygon $ABCDE$ is similar to the polygon $FGHKL$; the angle BAE is equal to the angle GFL : But it is E as BA to AE , so is G to FL : Therefore because ABE , FGL are two triangles hav-



ing one angle of the one equal to one angle of the other, and the sides about the equal angles proportional: [by 6. 6.] the triangle ABE will be equiangular to the triangle FGL : Therefore [by 4. 6.] it will be also similar to it; and so the angle ABE is equal to the angle FGL . But also the whole angle ABC is equal to the whole angle FGH , because the polygons are similar: Therefore the remaining angle EBC is equal to the remaining angle LGH . And because the triangles ABE , FGL are similar, it is as EB is to BA , so is LG to GF ; and because of the similar polygons AB is to BC , as FG is to GH ; it will be by equality, [by 22. 5.] as EB is to EC , so is LG to GH , viz. the sides proportional about the equal angles: Therefore [by 6. 6.] the triangle EBC is equiangular to the triangle LGH ; and also [by 4. 6.] similar to it. By the same reason the triangle ECD is similar to the triangle LHK . Therefore the polygons $ABCDE$, $FGHKL$ are divided into equal numbers of similar triangles.

I say also the triangles are homologous to the whole polygons: that is, the triangles are proportional, and the antecedents are ABE , EBC , ECD , and the consequents are FGL , LGH , LHK ; and one polygon $ABCDE$ has to the other $FGHKL$ a ratio being the duplicate of the ratio that one homologous side has to the other, that is of AB to FG .

For join AC , EH .

Then because the polygons are similar, the angle ABC is equal to the angle FGH , and it is as AB to BC , so is FG to GH : [by 6. 6.] the triangle ABC will be equiangular to the triangle FGH ; and so the angle BAC is equal to the angle $G FH$, and the angle BCA equal to the angle $G HF$. Moreover, because the angle BAM is equal to the angle GFN , and it has been proved that the angle ABM is equal to the angle FGN ; [by 32. 1.] the remaining angle AMB will be equal to the remaining angle ENG : Therefore the triangle ABM is equiangular to the triangle FCN . In like manner we prove that the triangle BMC is equiangular to the triangle GNH : Therefore [by 4. 6.] as AM is to MB , so is FN to NG , and as BM is to MC , so is GN to NH : Wherefore by equality [by 22. 5.] as AM is to MC , so is FN to NH . But as AM is to MC , so is the triangle ABM to the triangle MBC , and the triangle AME to the triangle EMC : For [by 1. 6.] they are

to one another as their bases: and [by 12. 5.] as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents: Therefore as the triangle AMB is to the triangle BMC , so is the triangle ABE to the triangle CBE . But as AMB is to BMC , so is AM to MC ; and therefore [by 11. 5.] as AM is to MC , so is the triangle ABE to the triangle EBC . By the same reason, as FN is to NH , so is the triangle FGL to the triangle GLH . But AM is to MC , as FN is to NH : Therefore [by 11. 5.] as the triangle ABE is to the triangle EBC , so is the triangle FGL to the triangle GLH ; and [by 16. 5.] as the triangle ABE is to the triangle FGL , so is the triangle EBC to the triangle GLH . After the like manner we demonstrate, by joining BD, GK , that the triangle EBC is to the triangle GLH , as the triangle ECD is to the triangle LHK . And because the triangle ABE is to the triangle FGL , as the triangle EBC is to the triangle GLH , and so is the triangle ECD to the triangle LHK ; it will be [by 12. 5.] as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents: Therefore as the triangle ABE is to the triangle FGL , so is the polygon $ABCDE$ to the polygon $FGHKL$. But the triangle ABE to the triangle FGL , has a ratio duplicate to that which the homologous side AB has to the homologous side FG : for [by 19. 6.] similar triangles are in the duplicate ratio of their homologous sides: therefore also the polygon $ABCDE$ to the polygon $FGHKL$ has a ratio duplicate to that which the homologous side AB of the one, has to the homologous side FG of the other.

Therefore similar polygons are [or may be] divided into equal numbers of similar triangles, homologous to the whole polygons; and one polygon to the other has a ratio which is the duplicate of the ratio that one homologous side of the one has to an homologous side of the other. Which was to be demonstrated.

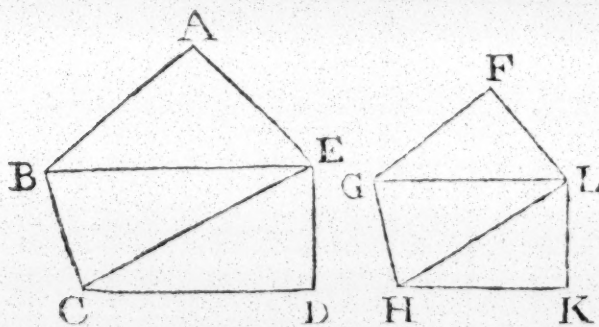
Corollary 1. We demonstrate after the same manner in similar quadrilateral figures, that they are to one another in the duplicate ratio of their homologous sides. This has been also proved of triangles, therefore universally similar right lined figures are to one another in the duplicate ratio of their homologous sides.

Corollary 2. And if a third proportional x be found to AB, FG : [by 10. def. 5.] AB will be to x , in the duplicate ratio of that of AB to FG : But the ratio of one polygon to the other, and of one quadrilateral figure to the other, is the duplicate of the ratio that one homologous side has to another; that is, of AB to FG : This is also demonstrated of triangles: therefore universally it is manifest, that if three right lines be proportional, as the first is to the third, so is any right lined figure described upon the first, to a similar right lined figure alike described upon the second.

Otherwise.

We shall demonstrate otherwise more expeditiously that the triangles are homologous.

For again let $ABCDE, FGHLK$ be the similar polygons; and join BE, EC, GL, LH : I say as the triangle ABE is to the triangle FGL , so is the triangle EBC to the triangle LGH , and the triangle CDE to the triangle HKL .



For because the triangle ABE is similar to the triangle FGL , the triangle ABE to the triangle FGL [by 19. 6.] will be in the duplicate ratio of BE

to GL . By the same reason the triangle EBC to the triangle LGH is in the duplicate ratio of BE to GL : then [by 11. 5.] as the triangle ABE is to the triangle FGL , so is the triangle EBC to the triangle LGH . Again, because the triangle EBC is similar to the triangle LGH ; [by 19. 6.] the triangle EBC has a ratio to the triangle LGH , the duplicate of the ratio of the right line CE to the right line HL . By the same reason also the triangle ECD has a ratio to the triangle LHK the duplicate of that of CE to HL : Therefore [by 11. 5.] as the triangle EBC is to the triangle LGH , so is the triangle ECD to the triangle LHK : But it has been proved that as the triangle EBC is to the triangle LGH , so is the triangle ABE to the triangle FGL : Therefore as the triangle ABE is to the triangle FGL , so is the triangle EBC to the triangle

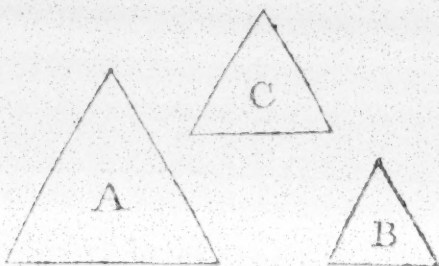
angle GLH , and the triangle ECD to the triangle LHK ; and therefore [by 12. 5.] as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, and the rest, as in the former demonstration. Which was to be demonstrated.

PROP. XXI. THEOR.

Right lined figures which are similar to the same right lined figure, are similar to one another.

Let the right lined figures A, B be each of them similar to the right lined figure C : I say the right lined figure A is also similar to the right lined figure B .

For because the right lined figure A is similar to the right lined figure C ; [by def. 1. 6.] it will be equiangular to it, and have the sides about the equal angles proportional. Again, because the right lined figure B is similar to the right lined figure C , it will be equiangular to it, and have the sides about the equal angles



proportional; therefore each of the figures A, B is equiangular to the right lined figure C ; and has the sides about the equal angles proportional: Wherefore the right lined figure A [by 1. ax.] is equiangular to B , and [by 11. 5.] has the sides about the equal angles proportional; and accordingly [by 1. def. 6.] A is similar to B . Which was to be demonstrated.

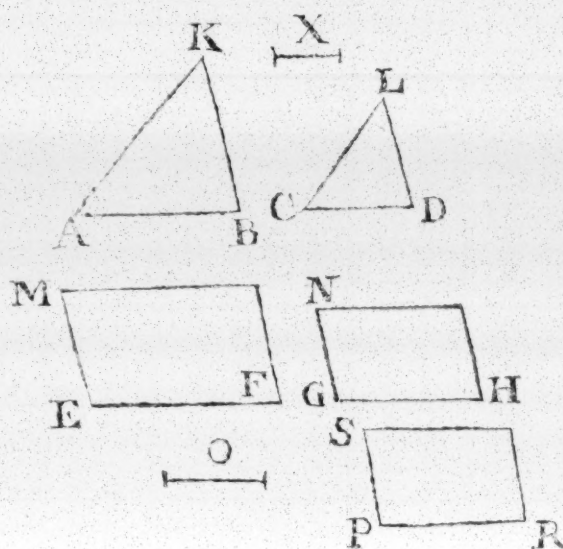
PROP. XXII. THEOR.

If four right lines be proportional, the right lined figures that are described upon them similar and alike situate will be also proportional: and if right lined figures described upon four right lines, similar and alike situate be proportional; those right lines will be proportional.

Let the four right lines AB, CD, EF, GH be proportional, viz. as AB is to CD , so is EF to GH ; and upon AB, CD , let the right lined figures KAB, LCD similar and

and alike situate be described; but upon EF , GH let the right lined figures MF , NH similar and alike situate, be described: I say as the right lined figure KAB is to the right lined figure LCD , so is the right lined figure MF to the right lined figure NH .

For [by 11. 6.] find a third proportional x to AB , CD ; and a third proportional o to EF , GH . And because AB



is to CD , as EF is to GH . And as CD is to x , so is GH to o : it will be, by equality [by 22. 5.] as AB is to x , so is EF to o . But [by 2. cor. 20. 6.] as AB is to x , so is the right lined figure KAB to the right lined figure LCD . But as EF is to o , so is the right lined figure

MF , to the right lined figure NH : Therefore [by 11. 5.] as KAB is to LCD , so is MF to NH .

But now let the right lined figure KAB be to the right lined figure LCD , as the right lined figure MF is to the right lined figure NH : I say as AB is to CD , so will EF be to GH .

For [by 12. 6.] make as AB is to CD , so is EF to PR . And [by 18. 6.] upon PR , describe the right lined figure SR similar and alike situate to either of the right lined figures MF or NH .

Then because AB is to CD , as EF is to PR , and upon AB , CD are described the right lined figures KAB , LCD similar and alike situate. But upon EF , PR the right lined figures MF , SR , similar and alike situate. [by the first part of this] it is as the right lined figure KAB is to the right lined figure LCD , so is the right lined figure MF , to the right lined figure SR . But [by supposition] as the right lined figure KAB , is to the right lined figure LCD , so is the right lined figure MF to the right lined figure NH : Therefore [by 11. 5.] MF has the same ratio to each of the right lined figures NH , SR : Consequently

quently [by 9. 5.] the right lined figures NH , SR are equal: They are also similar and alike situate: Therefore [by the following lemma] GH is equal to PR . And because AB is to CD , as EF is to PR , and PR is equal to GH : [by 7. 5.] AB will be to CD , as EF is to GH .

If therefore four right lines be proportional, the right lined figures that are described upon them similar and alike situate, will be also proportional: and if right lined figures described upon four right lines similar and alike situate, be proportional; these right lines will be proportional. Which was to be demonstrated.

LEMMA.

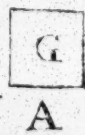
But we shall thus demonstrate, that the homologous sides of similar and equal right lined figures, are equal.

Let the right lined figures NH , SR be similar and equal: and let HG be to GN , as RP is to PS : I say RP is equal to HG .

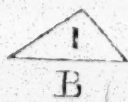
For if they be unequal, one of them will be the greater. Let this be RP . Then because RP is to PS , as HG is to GN : and alternately [by 16. 5.] as RP is to HG , so is PS to GN . But PR is greater than GH ; therefore PS will be greater than GN : Wherefore [by 20. 6.] the right lined figure RS is greater than the right lined figure HN ; and also equal; which cannot be: Therefore PR is not unequal to GH : Wherefore it is equal to it. Which was to be demonstrated.

This proposition is always true; if all the four figures be similar and alike situate. But when only two and two are so: it will not always be true, unless the two similar and alike situate figures be both described upon those two right lines that are the antecedents and consequents of the equal ratios; as if the right line A be to B , as C is to D , and the figures G , H described upon the first and fourth right lines A and D be similar and alike situate, and

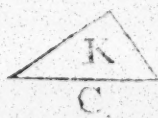
both the figures I , K described upon the second and



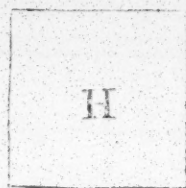
A



B



C



D

third right lines B , C be both similar and alike situate, these four figures G , I , K , L will not be proportional, as is easily apprehended, from the bare contemplation of the figures.

Should

Should not the lemma to this proposition have been made a proposition, and set down before this proposition?

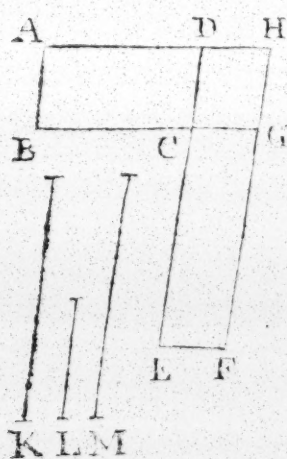
PROP. XXIII. THEOR.

Equiangular parallelograms are to one another in a ratio compounded of the ratios of their sides^k.

Let AC , CF be equiangular parallelograms, having the angle BCD equal to the angle ECG : I say the parallelogram AC is to the parallelogram CF in the ratio compounded of the ratios of the sides: that is, of the ratio of BC to CG , and of the ratio of DC to CE .

For put BC , CG in the same right line. Then [by 14. 1.] DC will be in the same right line with CE : and compleat the parallelogram DG , also let K be any right line, and [by 22. 6.] make as BC to CG , so is K to L , and as DC is to CE , so is L to M .

Then the ratios of K to L , and L to M are the same as the ratios of the sides, *viz.* that of BC to CG , and of DC to CE : But [by 5. def. 6.] the ratio of K to M is compounded of the ratio of K to L , and of the ratio of L to M .



Wherefore the ratio of K to M is the ratio compounded of the ratios of the sides. And because [by 1. 6.] as BC is to CG , so is the parallelogram AC to the parallelogram CH : But as BC to CG , so is K to L : Therefore [by 11. 5.] as K is to L , so is the parallelogram AC to the parallelogram CH . Again, because [by 1. 6.] as DC is to CE , so is the parallelogram CH to the parallelogram CF ; and as DC is to CE , so is L to M : Therefore as L is to M , so

[by 11. 5.] will the parallelogram CH be to the parallelogram CF . And so since it has been proved that as K is to L , so is the parallelogram AC to the parallelogram CH : But as L to M , so is the parallelogram CH to the parallelogram CF : it will be, by equality [by 22. 5.] as K is to M , so is the parallelogram AC to the parallelogram CF . But the ratio of K to M is that compounded of the ratios of the sides: Therefore the ratio of the parallelogram AC

to the parallelogram $c F$ is compounded of the ratios of their sides.

Therefore equiangular parallelograms are to one another in a ratio compounded of the ratios of their sides. Which was to be demonstrated.

* Some may perhaps like the following demonstration of this proposition better than Euclid's.

It will be [by 1. 6.] as $B C$ to $c G$, so is the parallelogram $A C$ to the parallelogram $c H$. And as $B C$ is to $c G$, so is the rectangle under $B C$ and $c D$ to the rectangle under $c G$ and $c D$. Wherefore [by equality] as the rectangle under $B C$ and $c D$ is to the rectangle under $c G$ and $c D$, so is the parallelogram $A C$ to the parallelogram $c H$.

Therefore [by alternation] as the rectangle under $B C$ and $c D$, is to the parallelogram $A C$, so is the rectangle under $c G$ and $c D$, to the parallelogram $c H$. For the same reason, the rectangle under $c E$ and $c G$ is to the parallelogram $c F$, as the rectangle under $c D$ and $c G$ is to the parallelogram $c H$. Wherefore [by equality] as the rectangle under $B C$ and $c D$, is to the parallelogram $A C$, so is the rectangle under $c E$ and $c G$ to the parallelogram $c F$. Therefore [by alternation] as the rectangle under $B C$ and $c D$ is to the rectangle under $c E$ and $c G$, so is the parallelogram $A C$, to the parallelogram $c F$. W. W. D.

If there be two triangles having one angle of the one equal to one angle of the other: those triangles will be to one another in the ratio compounded of the ratios of the sides containing the equal angles.

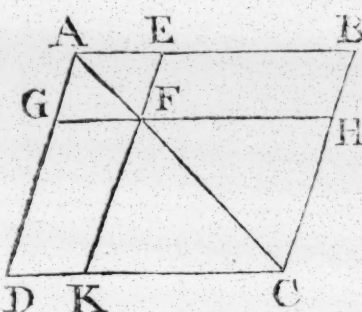
This will appear evident by supposing the angles $B, D; E, G$ of the equiangular parallelograms $A B C D, E C G F$ to be joined by right lines. For then because [by 34. 1.] the triangles $B D C, E C G$ are each the one half of the said parallelograms; since halves are as the wholes, and since the vertical angles of these triangles at c are equal to one another. Therefore [by this prop.] the two triangles $B D C, E C D$ will be to one another in a ratio compounded of the ratios of their sides: and so [by def. 5. 6.] one triangle will be to another, as the rectangle under the sides containing an angle of the one, is to the rectangle under the sides containing an equal angle of the other.

This theorem is also easily demonstrated. If the diagonals of two trapeziums intersect one another in a given angle, those trapeziums will be to one another in a ratio compounded of the ratios of the diagonals.

PROP. XXIV. THEOR.

The parallelograms that stand about the diameter of every parallelogram, are similar to one another, and to the whole parallelogram¹.

Let $ABCD$ be a parallelogram, whose diameter is AC , and let EG, HK be parallelograms standing about the dia-



meter AC : I say the parallelograms EG, HK are similar to one another, and to the whole parallelogram $ABCD$.

For because EF is drawn parallel to one side BC of the triangle ABC ; [by 2. 6.] it will be as BE to EA , so is CF to FA . Again, because FG is drawn parallel to the side CD of the triangle ACD , as CF is to FA , so will DG be to GA . But it has been proved that as CF is to FA , so is BE to EA : Therefore [by 11. 5.] as BE is to EA , so is DG to GA : and by compounding [by 18. 5.] as BA is to AE , so is DA to AG ; and inversely [by 16. 5.] as BA is to AD , so is AE to AG : Therefore the sides of the parallelograms $ABCD, EG$ about the common angle BAD are proportional. And because GF is parallel to DC , the angle AGF [by 29. 1.] is equal to the angle ADC . But the angle GFA is equal to the angle DCA , and the angle DAC is common to each of the triangles ADC, AGF : Therefore the triangle ADC will be equiangular to the triangle AGF . By the same reason the triangle ABC will be equiangular to the triangle AFE : Therefore the whole parallelogram $ABCD$ is equiangular to the parallelogram EG : Wherefore [by 4. 6.] as AD is to DC , so is AG to GF . But as DC is to CA , so is CF to FA , and as AC is to CB , so is AF to FE ; and also as CB is to BA , so is FE to EA : Wherefore because it has been proved that DC is to CA , as GF is to FA , and as AC is to CB , so is AF to FE ; it will be, by equality, [by 22. 5.] as DC is to CB , so is GF to FE : Therefore the sides of the parallelograms $ABCD, EG$ which are about the equal angles, are proportional; and accordingly [by 1. def. 6.] the parallelogram $ABCD$ is similar to the parallelogram EG . By

the same reason the parallelogram $A B C D$ is also similar to the parallelogram $K H$: Therefore the parallelograms $E G$, $H K$ are each of them similar to the parallelogram $A B C D$. But [by 21. 6.] those right lined figures which are similar to the same right lined figure, are similar to one another: Therefore the parallelogram $E G$ is similar to the parallelogram $H K$.

Wherefore the parallelograms that stand about the diameter of every parallelogram are similar to one another, and to the whole parallelogram. Which was to be demonstrated.

Some have observed here that the parallelograms about the diameter are similar figures, having their sides directly proportional, and the complements are equal parallelograms, having their sides reciprocally proportional to one another: Also either of the complements is a mean proportional between the parallelograms about the diameter. And these are to one another in the duplicate ratio of their homologous sides.

PROP. XXV. PROBL.

To describe a right lined figure similar to one given right lined figure, and equal to another.

Let $A B C$ be the given right lined figure to which the figure is to be made similar, and D the right lined figure to which it is to be made equal: it is required to make a right lined figure similar to $A B C$, and equal to D .

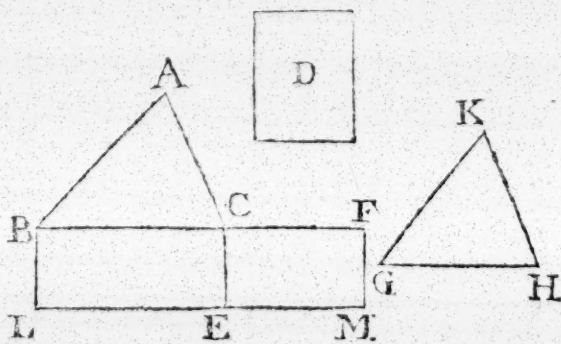
For [by 44. and 45. 1.] to the right line $B C$ apply the parallelogram $B E$ equal to the triangle $A B C$, and to the right line $C E$ apply the parallelogram $C M$ equal to D , and having the angle $F C E$, equal to the angle $C B L$: Then

[by 14. 1.] $B C$, $C E$ are both in one right line, as also $L E$, $E M$.

Now [by 13. 6.] find a mean proportional $G H$ between

$B C$, $C E$, and [by 13. 6.] upon $G H$

describe the right lined figure $K G H$



similar

similar to the right lined figure ABC , and alike situate.

Then because it is as BC to GH , so is GH to CF , and if three right lines be proportional, as the first is to the third, so [by 2. cor. 20. 6.] is the right lined figure described upon the first to a similar right lined figure alike situated described upon the second: Therefore it will be as BC is to CF , so is the triangle ABC , to the triangle KGH . But [by 1. 6.] as BC is to CF , so is the parallelogram BE to the parallelogram EF : And therefore as the triangle ABC is to the triangle KGH , so is the parallelogram BE to the parallelogram EF . Wherefore alternately [by 16. 5] as the triangle ABC is to the parallelogram BE , so is the triangle KGH to the parallelogram EF . But [by constr.] the triangle ABC is equal to the parallelogram BE : Therefore the triangle KGH also is equal to the parallelogram EF . But the parallelogram EF is equal to the right lined figure D . Therefore the triangle KGH is equal to the right lined figure D . But [by constr.] KGH is similar to the triangle ABC .

Therefore the right lined figure KGH is described similar to the given right lined figure ABC , and equal to the given right lined figure D . Which was to be done.

PROP. XXVI. THEOR.

If a parallelogram be taken away from a parallelogram similar to the whole, alike situate, and both having one common angle: I say they will also both have the same common diameter.

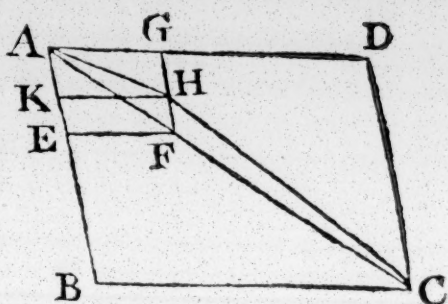
For from the parallelogram $ABCD$ let the parallelogram $AEFG$ be taken, similar to $ABCD$, alike situate, and both having the same common angle DAB : I say both the parallelograms $ABCD$, $AEFG$ will have the same common diameter.

For if not, let if possible, AHC be the diameter of them, and thro' H draw HK parallel to AD or BC .

Then because the parallelograms $ABCD$, KG have both the same diameter: [by 24. 6.] the parallelogram

$ABCD$

$ABCD$ will be similar to the parallelogram KG : Wherefore [by I. def. 6.] as DA is to AB , so is GA to AK . But because of the similar parallelograms $ABCD, EG$, as DA is to AB , so is GA to AE : and therefore [by II. 5.] as



GA is to AE , so is GA to AK : and so GA has the same ratio to AK , or AE : Wherefore [by 9. 5.] AE is equal to AK , a less equal to a greater; which is absurd. Therefore the parallelograms $ABCD, KG$ have not both the same common diameter. Wherefore the parallelograms $ABCD, AEFG$ have both the same common diameter.

If therefore a parallelogram be taken away from a parallelogram, similar to the whole, alike situate, and both having one common angle: they will also both have one common diameter. Which was to be demonstrated.

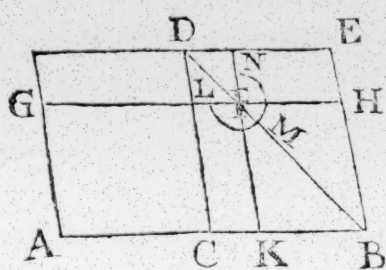
PROP. XXVII. THEOR.

The greatest of all parallelograms applied to the same right line deficient by parallelograms, similar and alike situate to that applied to half the line, is that applied to the half line, being similar to the deficient parallelograms applied to the other half line^m.

Let the right line be AB , which is bisected in c ; and to AB let the parallelogram AD be applied less than the parallelogram AE by the parallelogram CE similar and alike situate to the parallelogram described upon the line CB , viz. the one half of the right line AB : I say AD is the greatest of all the parallelograms applied to the right line AB , and deficient by parallelograms, similar and alike situate to CE . For apply the parallelogram AF to the right line AB , deficient in its figure by the parallelogram KH , similar to CE , and alike situate. I say the parallelogram AD is greater than the parallelogram AF .

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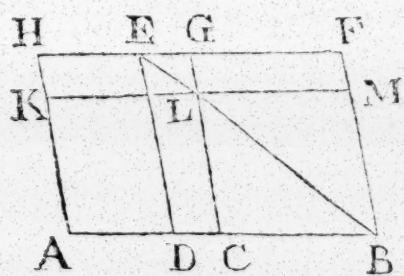
For



For because the parallelogram CE is similar to the parallelogram KH [by 26. 6.] they both stand about the same diameter: Draw the diameter DB , and describe the figure. Then because [by 43. 1.] CF is equal to FE , add KH which is common: Therefore the whole CH is equal to the whole KE . But [by 36. 1.] CH is equal to CG , because the right line AC is equal to CB : Therefore GC will be equal to EK . Add CF which is common, and the whole AF , is equal to the gnomon LMN ; wherefore CE , that is [by 36. 1.] the parallelogram AD is greater than the parallelogram AF .

* Let again the right line AB be bisected in C ; and let AL be applied deficient by the parallelogram CM . And again let the parallelogram AE be applied to the right line AB , deficient by the parallelogram DF , being similar and alike situate to the parallelogram CM , applied to the half line CB : I say the parallelogram AL applied to the half line is greater than the parallelogram AE .

For because DF is similar to CM ; [by 26. 6.] they are both about the same diameter. Let EB be their diameter, and describe the figure.



Then because [by 36. 1.] LF is equal to LH , for FG is equal to GH : LF will be greater than EK . But [by 43. 1.] LF is equal to DL : Therefore DL is greater than EK . Add KD which is common: then the whole AL is greater than the whole AE .

Therefore the greatest of all the parallelograms applied to the same right line deficient by parallelograms similar

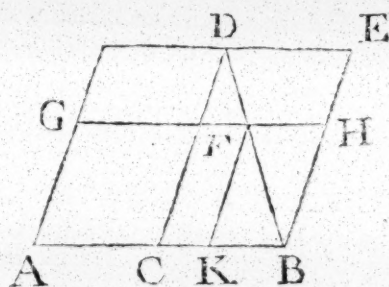
* This is the second case of this theorem (viz. when AD is less than AC , since the first is that when AD is greater than AC) and there is no other demonstration of the proposition, given by Euclid himself; this is therefore supplied to make the conclusion general by some of the ancients, as Theon, &c.

and

and alike situate to that applied to one half the line, is that applied to half the line, being similar to the deficient parallelogram applied to the other half line. Which was to be demonstrated.

^m Many have complained that this proposition is obscurely worded: and I must confess it is scarcely to be understood till the demonstration is read. Euclid certainly knew how to word it better. But as he could not have done it in so few words, he chose to give it as it is, rather than appear tedious. However the following attempt of mine to word it differently, is thus: *If there be any parallelogram applied to a given right line (that is, described upon it) and another of the same altitude equiangular to it be applied to half of that right line; and if any other parallelogram whatsoever of a lesser altitude equiangular to either of them be applied to that right line, and from this last parallelogram be cut off another parallelogram similar to that applied to half of the given right line, the parallelogram applied to half of the given right line will be the greatest of any one of these remaining parallelograms.* As let AB be the given right line,

AE the parallelogram applied to AB , and AD or CE the equiangular parallelogram of the same altitude applied to AC or CB one half of AB . And let AH be any parallelogram at pleasure applied to AB equiangular to AE , and having a lesser altitude. And from this parallelogram let the parallelogram KH be cut off similar to AD or CE . Then will



AD or CE be greater than the remaining parallelogram AF , and this is what Euclid has demonstrated in this proposition.

This theorem is very useful in the consideration of maximums and minimums.

PROP. XXVIII. PROBL.

To apply a parallelogram to a given right line equal to a given right lined figure, being deficient by a parallelogram, similar to a given parallelogram: But the given right lined figure to which the parallelogram is to be applied equal, must not be greater than that applied parallelogram which is described upon half the line; both the deficient par-

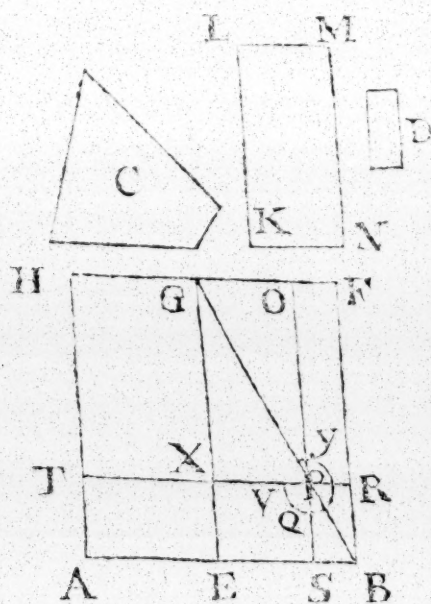
T +

allelograms

allelograms being similar to the given parallelogram, viz. the parallelogram being the deficiency of the parallelogram described upon the half line, and the parallelogram being the deficiency of the parallelogram required to be applied to which the deficient parallelogram is to be similar.

Let the given right line be AB , and c the given right lined figure to which the parallelogram applied to AB must be equal; which must not be greater than a parallelogram applied to the half of AB , the parallelograms being the deficiencies of these parallelograms being similar. And let D be the parallelogram to which the deficient parallelogram is to be similar: it is required to apply a parallelogram to the given right line AB equal to the given right lined figure c , deficient by a parallelogram similar to D .

Divide [by 10. 1.] AB in half at E , and [by 18. 6.] upon EB describe $EBFG$ similar to D , and alike situate, and compleat the parallelogram AG : then AG is either equal to c , or greater than it, by reason of the limitation.



And if AG be equal to c , what was required will be done; for the parallelogram AG is applied to the right line AB , equal to the given right lined figure c , deficient by the parallelogram EF similar to the figure D . But if it be not equal, HE will be greater than c . But HE is equal to EF : therefore EF is greater than c . Now let a parallelogram $KLMN$ be made [by 25. 6.] equal to the excess of EF above c , but similar and alike situated to D : But D is similar to EF : Wherefore KM will be also similar to EF . Therefore let the right line LK be homologous to GE , and LM to GF . And because EF is equal to c and KM together: Therefore EF is greater than KM : Therefore [by cor. 1. 20. 6.] GE is greater than KL , and GF than ML . [By 3. 1.] draw GK equal to

to LK , and GO equal to LM , and [by 31. 1.] compleat the parallelogram $XGOP$: Then [by 24. 6.] XO is equal and similar to KM . But KM is similar to EF : Therefore [by 21. 6.] HO also is similar to EF : Wherefore [by 26. 6.] XO and EF both stand about the same diameter. Let GPB be their diameter, and compleat the figure.

Then because EF is equal to c , KM together; but XO is equal to KM , the remaining gnomon VQY , is equal to the remaining right lined figure c . And because [by 43. 1.] OR is equal to XS , add RS , which is common; then the whole OB , is equal to the whole XB . But XB [by 36. 1.] is equal to TE : Because the side AE is equal to the side EB : Wherefore TE is equal to OB . Add sx , which is common. Then the whole TS is equal to the whole gnomon VQY . But it has been proved that the gnomon VQY is equal to the right lined figure c : Therefore TS will be also equal to the right lined figure c .

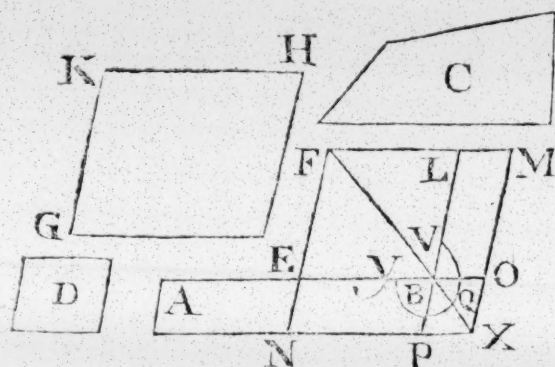
Therefore the parallelogram TS is applied to the given right line AB equal to the given right lined figure c , and deficient by the parallelogram RS similar to D , because RS is similar to OX . Which was to be done.

PROP. XXIX. PROBL.

To apply a parallelogram to a given right line equal to a given right lined figure, redundant by a parallelogram, similar to a given parallelogram."

Let the given right line be AB , and let c be the given right lined figure to which the parallelogram applied to AB is to be equal, and let D be the right lined figure to which the redundant parallelogram is to be similar: It is required to apply a parallelogram to the right line AB equal to the given right lined figure c , redundant by a parallelogram similar to D .

Bisect AB [by 9. 1.] in E , and [by 18. 6.] describe the parallelogram EL upon the right line EB similar to D , and alike situate: also make [by 25. 6.] the parallelogram GH equal to EL , c both together, but similar to D , and alike situate: Then GH is similar to EL . Let KE , FL be homologous sides, as also KG , FE : And because the parallelogram GH is greater than the parallelogram EL ,
the



the right line KH will be greater than the right line FL , and KG greater than FE . Produce FL , FE , and [by 3. 1.] make FLM equal to KH , and FEN equal to KG ; and compleat the parallelogram MN :

Then MN is equal and similar to GH : But GH is similar to EL : Therefore [by 21. 6.] MN is also similar to EL ; and accordingly [by 26. 6.] EL , MN stand both about the same diameter. Draw their diameter FX , and describe the figure.

Then because GH is equal to EL , C together; but GH is also equal to MN ; MN will be equal to EL , C together: Take away EL , which is common; then the gnomon VQY is equal to the right lined figure C . And because EA is equal to EB ; [by 36. 1.] the parallelogram AN will be equal to the parallelogram NB ; that is [by 43. 1.] to LO : Add EX , which is common; then will the whole AX be equal to the gnomon VQY . But the gnomon VQY is equal to C : Therefore AX is also equal to C .

Wherefore the parallelogram AX is applied to the given right line AB equal to the given right lined figure C , redundant by the parallelogram PO similar to D , because EL is also similar to OP . Which was to be done.

" The twenty-eighth and twenty-ninth propositions are as obscurely worded as the twenty-seventh; but it is easy to alter them as I have done Euclid's twenty-seventh, in my note upon it.

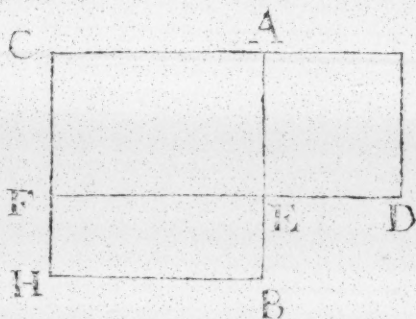
Altho' these problems are not of so general use in geometry, as many other propositions, yet Euclid was obliged to put them down, because he could not have demonstrated the tenth book without them.

PROP. XXX. THEOR.

To cut a given right line into mean and extreme ratio^o.

Let AB be the given right line. It is required to cut the same into mean and extreme ratio.

For [by 46. 1.] upon AB describe the square BC ; and [by 29. 6.] apply the parallelogram CD to AC equal to BC redundant by the parallelogram AD similar to BC . But BC is a square: Therefore AD will also be a square. And because BC is equal to



CD , take away CE , which is common: Then the remainder BF is equal to the remainder AD . But it is also equiangular to it: Wherefore [by 14. 6.] the sides of BF , AD about the equal angles are reciprocally proportional: Therefore as FE is to ED , so is AE to EB . But [by 34. 1.] FE is equal to AC , that is, to AB , and ED equal to AE . Wherefore as AB is to AE , so is AE to EB . But AB is greater than AE : Therefore AE is greater than EB .

Therefore the right line AB is cut into mean and extreme ratio in E , the greater segment being AE . Which was to be done.

Otherwise.

Let the given right line be AB . It is required to cut the same into mean and extreme ratio.

For [by 11. 2.] divide AB so in C , that the rectangle under AB , BC be equal to the square of AC .



Then because the rectangle under AB , BC is equal to the square of AC ; [by 17. 6.] it will be as AB to AC , so is AC to CB . Therefore [by def. 3. 6.] AB is cut into mean and extreme ratio. Which was to be done.

* This problem in fact is only a repetition of the eleventh proposition of the second book in other words. The first manner of solution here being indeed more difficult than that in the second book, and the latter the very same as that in the second book.

PROP

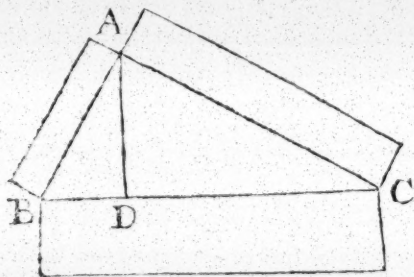
PROP. XXXI. THEOR.

In right angled triangles the figure described upon the side opposite to the right angle is equal to both the figures described upon the sides containing the right angle, which are similar to it, and have the same situation.

Let the right angled triangle be ABC , having the right angle BAC : I say the figure described upon BC is equal to those described upon BA , AC being similar to it, and having the same situation.

Draw the perpendicular AD .

Then because AD is drawn from the right angle A of the right angled triangle ABC perpendicular to the base BC : [by 8. 6.] the triangles ABD , ADC are both similar to one another, and to the whole triangle ABC : There-



fore it will be as CB is to BA , so is AB to BD : But when three right lines are proportional; as the first is to the third, so [by 2. cor. 20. 6.] is the figure described upon the first to a figure similar and alike situate,

described upon the second: Therefore as CB is to BD , so is the figure described upon CB to a similar and alike situate figure described upon BA . By the same reason, as BC is to CD , so is the figure described upon BC , to the figure described upon CA : Therefore as BC is to BD and DC together, so is the figure described upon BC to both the figures described upon BA , AC being similar and alike situate to the figure described upon BC . But BC is equal to BD , DC together: Therefore the figure described upon BC is equal to both the figures together which are described upon BA , AC similar and alike situate to the figure described upon BC .

Therefore in right angled triangles, the figure described upon the side opposite to the right angle, is equal to both the figures together described upon the sides containing the right angle, which are similar to it, and have the same situation. Which was to be demonstrated.

Otherwise.

Otherwise.

Because [by 23. 6.] similar figures are to one another in the duplicate ratio of their homologous sides; the figure described upon BC to the figure described upon BA will be in the duplicate ratio of that of BC to BA . But [by 1. cor. 20. 6.] the square of BC to the square of BA is in the duplicate ratio of BC to BA : Therefore [by 11. 5.] as the figure described upon BC is to the figure described upon BA , so is the square of BC to the square of BA . Also by the same reason as the figure described upon BC is to the figure described upon CD , so is the square of BC to the square of CA : and therefore the figure described upon BC , is to the figures described upon BA , AC together, as the square of BC is to the squares of BA , AC together. But the square of BC is [by 47. 1.] equal to the squares of BA , AC together: Therefore the figure described upon BC is equal to both the figures described upon BA , AC being similar to that described upon BC , and having the same situation. Which was to be demonstrated.

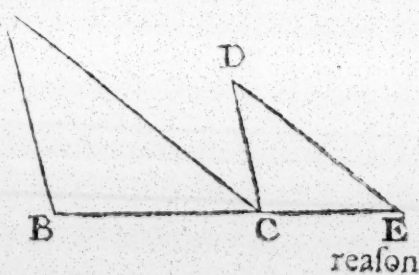
P This proposition is much more general than the forty-seventh of the first book; it extending to all similar figures whatsoever; whereas that is only confined to squares.

PROP. XXXII. THEOR.

If two triangles having two sides of the one proportional to two sides of the other, be so set together at one angle so that the homologous sides be parallel: the remaining sides of those triangles will be both in one right line.

Let there be two triangles ABC , DCE having two sides BA , AC of the one proportional to two sides CD , DE of the other, viz. as BA is to AC , so is CD to DE , and let AB be parallel to DC , and AC to DE : I say BC , CE will be both in the same right line.

For because AB is parallel to DC , and the right line AC falls upon them; the alternate angles BAC , ACD [by 29. 1.] will be equal to one another. By the same



reason the angle CDE is equal to the angle ACD : Wherefore the angle BAC is equal to the angle CDE . And because ABC, DCE are two triangles, having one angle A of the one equal to one angle D of the other, and the sides about the equal angles proportional, *viz.* as BA is to AC , so is CD to DE : the triangle ABC [by 6. 6.] will be equiangular to the triangle DCE : therefore the angle ABC is equal to the angle DCE . But the angle ACD has been proved to be equal to the angle BAC : Therefore the whole angle ACE is equal to both the angles ABC, BAC together. Add the common angle ACB : then the angles ACE, ACB together are equal to the angles BAC, ACB, ABC together. But [by 32. 1.] the angles BAC, ACB, ABC together, are equal to two right angles: and therefore the angles ACE, ACB together will be also equal to two right angles. And so two right lines BC, CE drawn contrariways from the point C in the right line AC make the adjoining angles ACE, ACB at the right line AC , equal to two right angles: Therefore [by 14. 1.] BC, CE will both lie in the same right line.

If therefore two triangles having two sides of the one proportional to two sides of the other, be so set together at one angle, that their homologous sides be parallel: the remaining sides of those triangles will be both in the same right line. Which was to be demonstrated.

PROP. XXXIII. THEOR.

In equal circles the angles have the same ratio, as the parts of the circumference on which they stand, whether those angles be at the centres, or at the circumferences: and moreover so are the sectors, as being at the centre^a.

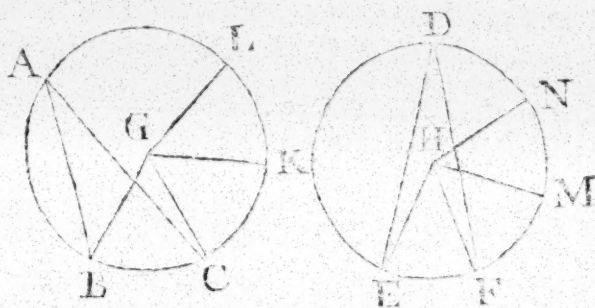
Let the equal circles be ABC, DEF , and let the angles BGC, EHF be at their centres G, H , and the angles BAC, EDF at their circumferences: I say as the part BC of the circumference is to the part EF , so is the angle BGC to the angle EHF ; the angle BAC to the angle EDF ; and moreover so is the sector BGC to the sector EHF .

For take orderly any number of parts CK, KL of the circumference of one circle each equal to the part BC , and an equal number FM, MN of any other parts of the circum-

circumference of the other circle each equal to EF ; and join GK , GL , HM , HN .

Then because the parts BC , CK , KL of the circumference of one circle are equal to one another, the angles BGC , CGK , KGL [by 27. 3.] will be equal to one another: Therefore the part BL of the circumference of one circle is the same multiple of the part BC of the circumference of that circle, as the angle BGL is of the angle BGC . And by the same reason the part EN of the circumference of the other circle is the same multiple of the part EF of the circumference, as the angle EHN is of the angle EHF . And if the part BL of the circumference be equal to the part EN of the other circumference; the angle BGL [by 27. 3.] will be equal to the angle EHN . And if the part BL of the circumference be greater than the part EN ; the angle BGL will be greater than the angle EHN : and if less, less. There are therefore

four magnitudes, viz. the two parts BC , EF of the circumferences of the circles, and the two angles BGC , EHF , and there



are taken the equimultiples of the part BC of the circumference of one circle, and of the angle BGC , viz. the part BL of the circumference, and the angle BGL , and equimultiples of the part EF of the circumference of the other circle, and of the angle EHF , viz. the part EN of the circumference, and the angle EHN . And it has been demonstrated, that if the part BL of one circumference exceeds the part EN of the circumference of the other circle, the angle BGL also exceeds the angle EHN ; if BL be equal to EN , the angle BGL is equal to EHN : and if BL be less than EN ; the angle BGL is less than the angle EHN : Therefore [by 5. def. 5.] as the part BC of one circumference is to the part EF of the other, so is the angle BGC to the angle EHF . But [by 15. 5.] as the angle BGC is to the angle EHF , so is the angle BAC to the angle EDF . For [by 20. 3.] each of them is double to each of these; and therefore as the part BC of the one circum-

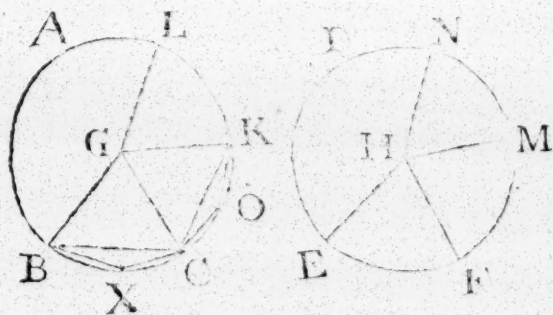
circumference, is to the part EF of the other, so is the angle BGC , to the angle EHF , and the angle BAC to the angle EDF .

Therefore in equal circles the angles have the same ratio, as the parts of their circumference upon which they stand, whether they stand at the centres or at the circumferences. Which was to be demonstrated.

I say moreover as the part BC of one circumference is to the part EF of the other, so is the sector GBC to the sector HEF .

For join BC , CK , and taking the points x , o , in the parts BC , CK of the circumferences. Join Bx , xC , Co , oK .

Then because the two sides BG , Gc are equal to the two sides cG , GK , and they contain equal angles; the base Bc will be equal to the base CK : Therefore [by 4. 1.] the triangle GBC also is equal to the triangle GCK . And because the part Bc of the circumference is equal to the part CK , and [by 3. ax.] the remaining part of one circumference, is equal to the remaining part of the other; therefore [by 27. 3.] the angle Bxc is equal to the angle CoK . Wherefore [by 11. def. 3.] the segment Bxc is similar to the segment CoK . And they are upon the equal right lines Bc , CK . But [by 24. 3.] those similar segments of circles that stand upon equal right lines, are themselves also equal: Therefore the segment Bxc is equal to the segment CoK .



But the triangle BcG also is equal to the triangle cGK : And therefore the whole sector GBC [by 2. ax.] will be equal to the whole sector GCK . By the same reason also the sector GKL is equal to each of the sectors GKC , GCB : Therefore the three sectors GBC , GCK , GKL are equal to one another. So likewise are the sectors HEF , HFM , HMN equal to one another: Therefore the part BL of the circumference is the same multiple of the part BC , as the sector GBL is of the sector GBC . By the same reason the part EN of the circumference is the same multiple of the part EF as the sector HEN is of the sector

sector HEF : and [from what has been already demonstrated] if the part BL of the circumference be equal to the part EN ; the sector GBL is also equal to the sector HEN : and if the part BL of it exceeds the part EN , the sector GBL also exceeds the sector HEN ; and if less, less. There are therefore four magnitudes, the two parts BC , EF of the circumferences, and the two sectors GBC , HEF ; and there are taken equimultiples of the part BC of the circumference and of the sector GBC , viz. the part BL of the circumference, and the sector GBL ; and equimultiples of the part EF of the circumference, and of the sector HEF , viz. the part EN of the circumference, and the sector HEN . And it has been proved if the part BL of the circumference exceeds the part EN ; the sector GBL will exceed the sector HEN ; and if BL be equal to EN ; the sector GBL is equal to the sector HEN ; and if the part BL of the circumference be less than the part EN , the sector GBL is less than the sector HEN . Therefore [by 5. def. 5.] as the part BC of one circumference is to the part EF of the other, so is the sector GBC to the sector HEF .

Corollary. It is also evident [by 11. 5.] that as one sector is to the other, so is one angle to the other.

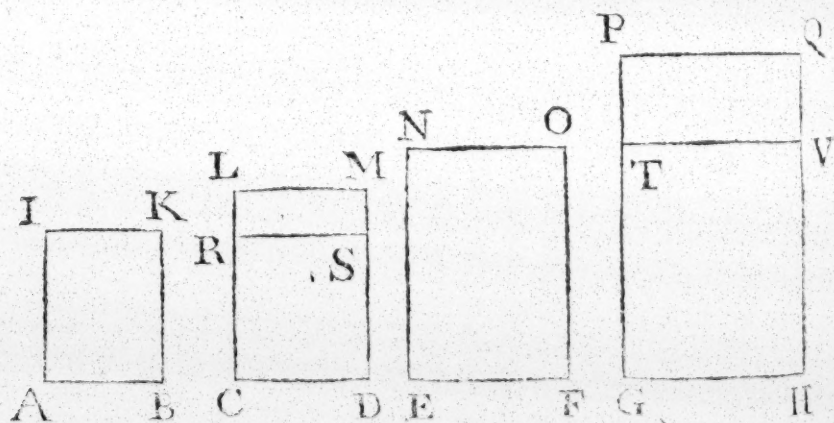
9 The demonstration of this proposition is another easy instance of the truth and excellence of Euclid's fifth definition of the fifth book. The proposition itself, or at least part of the demonstration, is thought to be Theon's, he himself saying, in his commentary upon Ptolemy's *Almagest*; *it is demonstrated by us in the edition of the Elements at the end of the sixth book, that sectors in equal circles are to sectors, as angles at the centres are to angles.*

Additions to the Sixth Book.

PROPOSITION I.

If four right lines be in the same proportion, the first to the second as the third to the fourth, as also four more; the fifth to the sixth as the seventh to the eighth. I say the rectangle under the first and fifth will be to the rectangle under the second and sixth, as the rectangle under the third and seventh is to the rectangle under the fourth and eighth.

Let there be four right lines AB, CD, EF, GH in the same proportion, as AB to CD , so is EF to GH : and



four more AI, CL, EN, GP , as AI to CL so is EN to GP . I say the rectangle under AB and AI will be to the rectangle under CD, CL , as the rectangle under EF, EN , is to the rectangle under GH, GP .

For make the rectangles AK, CL, EN, GP , under $AB, AI; CD, CL; EF, EN; GH, GP$ respectively. Make CR and DS equal to AI or BK , and draw RS . Also make GT or HV equal to EN or FO . And draw TV .

Then because [by supposition] it is as AB to CD , so is EF to GH ; and AI to CL , so is EN to GP . Therefore [by I. 6.] as the rectangle AK is to the rectangle cs ,

so will the rectangle EO be to the rectangle GV . And [alternately] as the rectangle AK is to the rectangle EO , so will the rectangle CS be to the rectangle GV . Also as the rectangle CS is to the rectangle CM , so will the rectangle GV be to the rectangle GQ [by 1. 6.] And [alternately] as the rectangle CS , is to the rectangle GV , so is the rectangle CM to the rectangle GQ . Therefore [by 11. 5.] as the rectangle AK is to the rectangle EO , so is the rectangle CM , to the rectangle GQ . And [inversely] as the rectangle AK is to the rectangle CM , so is the rectangle EO to the rectangle GQ ; that is, as the rectangle under AB and AI , is to the rectangle under CD and CL ; so is the rectangle under EF and EN , to the rectangle under GH and GP .

Therefore, &c. Which was to be demonstrated.

Otherwise.

The ratio of the rectangle AK is [by 23. 6.] to the rectangle CM compounded of the ratio of AB to CD , and of AI to CL . Also the ratio of the rectangle EO to the rectangle GQ is compounded of the ratio of EF to GH , and that of EN to GP . But the ratio of EF to GH [by supposition] is equal to the ratio of AB to CD , and the ratio of EN to GP equal to the ratio of AI to CL . But because those compound ratios which consist of equal numbers of equal simple ratios, are equal to one another; therefore the ratio of the rectangle AK to the rectangle CM , is equal to the ratio of the rectangle EO to the rectangle GQ . Wherefore the rectangles under AB , AI ; CD , CL ; EF , EN ; GH , GP will be proportional.

Therefore, &c. Which was to be demonstrated.

PROP. II. PROBL.

To inscribe a triangle within a given triangle, equiangular to a given triangle.

Let ABC be a given triangle, and DEF another. It is required to inscribe a triangle within the given triangle ABC , equiangular to the given triangle DEF .

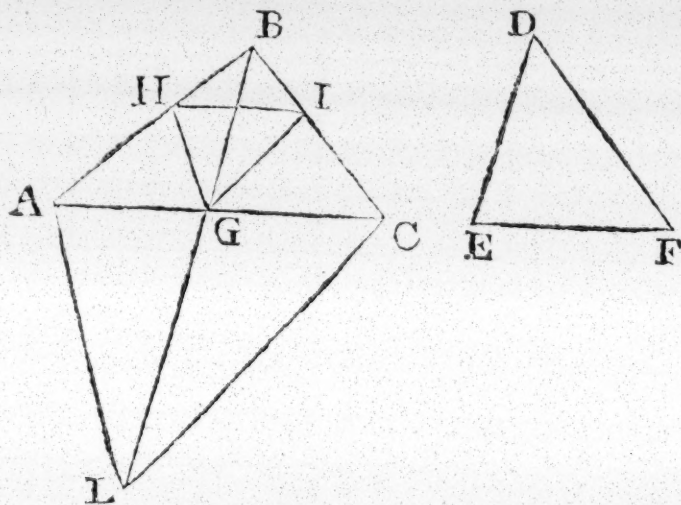
At the extremes A , C , of the given side AC , make [by 23. 1.] the angle ACL equal to the angle F of the given triangle DEF , and the angle LAC equal to the angle DEF of the given triangle DEF . And let the sides AL ,

U 2

CL

CL meet in the point L . Join BL cutting the side AC in the point G . From G draw [by 31. 1.] GH parallel to AL , and GI parallel to CL , meeting the sides AB , BC of the given triangle ABC in the points H , I , and join HI .

Then because [by constr.] GH is parallel to AL ; it will be [by 2. 6.] as AH is to HB , so is LG to GB . And because GI is parallel to LC , it will be as LG is to GB , so is IC to BI . And so [by 11. 5.] as AH is to HB , so will CI be to IB . Therefore [by 2. 6.] HI will be parallel to AC ; and so the angle BHI [by 29. 1.] will be equal to the angle BAG . Also because HG [by constr.] is parallel to AL , the angle BHG will be equal to the angle BAL ;



and so [taking equals from equals] there will remain the angle GHI equal to the angle LAG . But [by construction] the angle LAG is equal to the

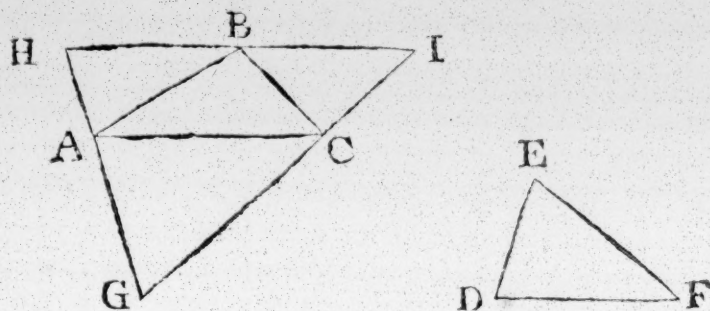
angle E of the triangle DEF . Therefore the angle GHI will be equal to the angle DEF . After the same manner it is demonstrated, that the angle HIG will be equal to the angle DFE of the given triangle DEF . Wherefore [by 32. 1.] the remaining angles HGI , EDF will be equal to one another. Therefore the triangle GHI will be equiangular to the given triangle DEF , and is inscribed in the given triangle ABC . Which was to be demonstrated.

PROP. III. PROBL.

To circumscribe a triangle about a given triangle, equiangular to a given triangle.

Let ABC be a given triangle, and DEF another. It is required to circumscribe a triangle about the given triangle ABC equiangular to the given triangle DEF .

Upon



Upon the base AC of the given triangle ABC , make [by 23. 1.] the triangle AGC equiangular to the given triangle DEF ; so that the angle GAC be equal to the angle EDF , and the angle ACG equal to the angle EFD , and the angle G equal to E . Continue out GA , GC , and through the point B draw [by 31. 1.] the right line HI parallel to the base AC , cutting the continuations of GA , GC in the points H , I ; then will the triangle GHI be circumscribed about the triangle ABC equiangular to the given triangle DEF .

For because [by construction] HI is parallel to AC , the angle GHI [by 29. 1.] will be equal to the angle GAC , that is, [by construction] to the angle EDF , and the angle GHI equal to the angle GCA , that is, to the angle EFD . Therefore [by 32. 1.] the remaining angles HGI , DEF will be equal to one another. And so the triangles GHI , DEF will be equiangular, and the triangle GHI is circumscribed about the given triangle ABC .

Therefore, *Q. E. D.* Which was to be demonstrated.

PROP. IV. PROBL.

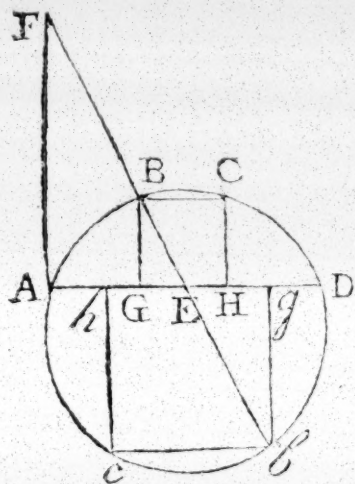
To inscribe a square in a given segment of a circle.

Let $ABCD$ be a segment of a circle, whose base is AD . It is required to inscribe a square in the segment $ABCD$ of a circle.

Bisect [by 10. 1.] the base AD of the segment in the point E ; and at one end A of the base of the segment, erect [by 11. 1.] the perpendicular AF , equal to the base AD , and draw a right line from E to F , cutting the circumference of the segment in the point B . From B to the base AD , let fall [by 12. 1.] the perpendicular BG . Make EH equal to GE . Then will GB and GH be each one side of the square required. Draw HC perpendi-

cular to AD , and join BC , and the square will be completed.

For because [by construction] BG is parallel to AF ; therefore [by 4. 6.] it will be as AE is to AF , so is GE to GB . But AF [by construction] is double to AE . Therefore GB will be double to GE ; and because [by construction] EH is equal to GE , and GB, HC are perpendicular to the base AD of the segment, these perpendiculars will be equal to one another (as is easily proved [by 35. 1.] by first drawing a diameter of the circle parallel to the base AD of the segment.) Wherefore [by 33. 1.] BC will be equal to GH . Conse-



quently the four right lines BG, GH, CH, BC are equal to one another; and since the angles BGH, GHC are right angles [by construction] and the opposite angles of every parallelogram [by 34. 1.] are equal to one another. Therefore will the angles BCH, GBC be right angles; and so the figure $GBCH$ will be a square; and it is inscribed in the given segment $ABCD$ of a circle.

Note, if the right line FB be continued to cut the circumference of the greater segment $Ac bD$ of the circle in the point b , and $b g$ be drawn from the point b perpendicular to the base AD of the segment. This will be one side of a square $c b g b$ inscribed in the greater segment of the circle, which square may be completed after the same manner as the square $GBCH$ has been in the lesser segment $ABCD$.

Note also, that by a like manner of construction, a square may be inscribed in a given sector of a circle.

PROP. V. PROBL.

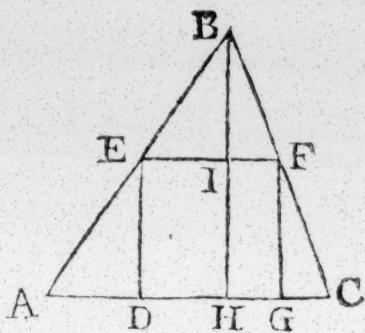
To inscribe a square in a given triangle.

Let there be a given triangle ABC . It is required to inscribe a square in the given triangle ABC .

From the angle B of the given triangle ABC , let fall [by 12. 1.] the perpendicular BH upon the base AC ; and

so

so divide [by 10. 6.] the perpendicular BH in I , that IH be to BI , as the base AC is to the perpendicular BH . And thro' I [by 31. 1.] draw the right line EF parallel to the base AC meeting the sides of the triangle in the points E, F . From E, F draw ED, FG parallel to the perpendicular BH ; then will the figure $DEFG$ be the square required.



For because the triangles ABC, EBF are equiangular, the right line EF [by construction] being parallel to AC , and so [by 29. 1.] the angles $BEF, BAC; BFE, BCA$ respectively equal to one another. And since the bases of equiangular triangles [by 4. 6. and 18. 5.] are proportional to their altitudes. Therefore it will be as AC is to BH , so is EF to BI . But [by construction] as AC is to BH , so is IH to BI . Therefore [by 11. 5.] EF will be to BI , as IH is to BI ; and therefore [by 9. 5.] EF will be equal to IH . And because [by construction] EF is parallel to AC , and ED, FG parallel to BH ; therefore [by 30. and 34. 1.] ED will be equal to FG , and each of these equal to IH , also EF will be equal to DG . Wherefore since EF, IH have been proved to be equal to one another; the four right lines DE, EF, FG, DG will be equal to one another; and [by 29. 1.] the angles D, G will be right angles; and [by 33. 1.] so will the angles E, F . Therefore the figure $DEFG$ will be a square, and it is inscribed in the given triangle ABC .

Therefore, &c. Which was to be done.

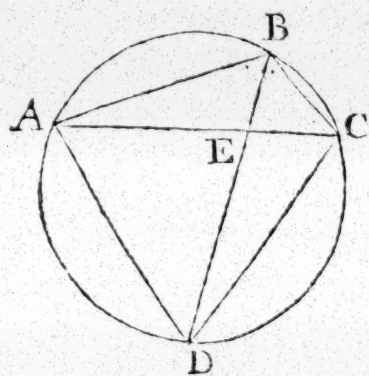
PROP. VI. THEOR.

If one angle of a triangle be bisected by a right line falling upon the base or side opposite to that angle, the rectangle under the two sides containing that angle will be equal to the square of the bisecting line, together with the rectangle under the segments of the base made by the falling of the bisecting line aforesaid upon it.

Let ABC be the given triangle; and from the angle B , opposite to the base AC , let the right line BE , bisecting the angle ABC , divide the base AC into two segments, AE , EC ; I say the rectangle under the sides AB , BC will be equal to the rectangle under the segments AE , EC of the base, together with the square of the bisecting right line BE .

For about the triangle ABC describe a circle [by 5. 4.] and continue down the right line BE to cut the circle in D ; and join AD , DC .

Then the triangles ABD , BCE will be equiangular, because the angles ABE , EBC are equal [by supposition], and the angles ADB , BCE in the same segment of the circle [by 21. 3.] will be equal too. And so [by 32. 1.] the remaining angles BAD , BEC are equal too. Therefore [by 4. 6.] as AB is to BD , so will BE be to BC .



But [by 35. 3.] the rectangle under AE and EC is equal to the rectangle under DE and EB . And [by 3. 2.] the rectangle under BD and BE is equal to the rectangle under DE and EB , together with the square of BE . Therefore the rectangle under BD and BE will be equal to the rectangle under AE and EC together with the square of EB ;

that is, because it has been proved that AB is to BD , as BE is to BC , and [by 16. 6.] the rectangle under BD , BE is equal to the rectangle under AB , and BC ; the rectangle under AB and BC will be equal to the rectangle under AE and EC , together with the square of EB .

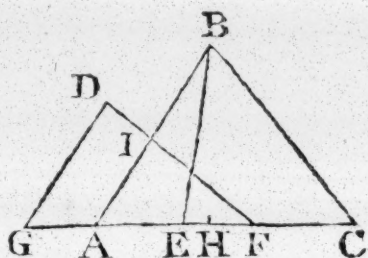
Therefore, &c. Which was to be demonstrated.

PROP. VII. THEOR.

If there be a given point D without a given triangle ABC , and from the vertical angle B a right line BE be drawn upon the base, dividing the said triangle into two parts ABE , EBC having a given ratio: and if the base AC be continued out meeting a right line DG drawn from D , parallel to the

the side AB , and GH be taken such, that GD be to AB , as AE is to AH . And if it be again made as AG is to AF , so is HF to AH ; and a right line DF be drawn from the given point D . This line will cut off a triangle AIF from the given triangle ABC equal to the triangle ABE .

For since [by construction] it is as GA is to AF , so is HF to AH , and [by 18. 5.] as GF is to AF , so is AF to AH ; and (because of the similar triangles GDF , AIF , for GD , AI are parallel) as GD is to GF , so is AI to AF ; therefore [by equality 11. 5] as GD is to AI , so is AF to AH (the ratio of GF to AF being common) that is [by supposition] as AB is to AI , so is AF to AE ; therefore the triangles ABE , AIF having one angle A common to both: have their sides reciprocally proportional; and so [by 15. 6.] they will be equal to one another; therefore as the triangle ABE is to the triangle ABC , that is [by 1. 6.] as AE is to EC , so will the triangle AIF be to the quadrilateral figure $FIEC$.



Therefore, &c. Which was to be demonstrated.

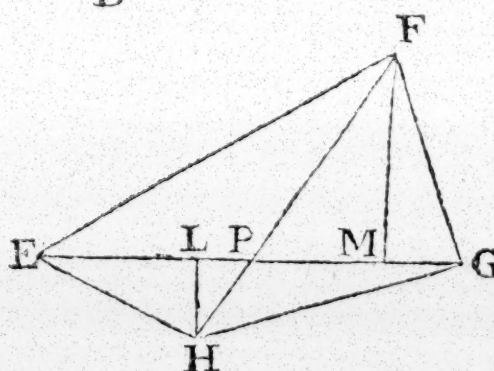
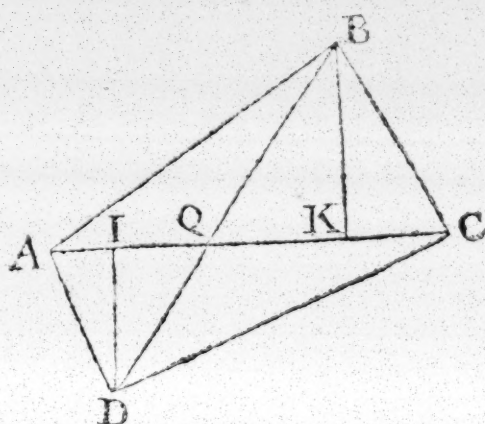
SCHOLIUM.

Hence if a given triangle AIF is to be cut off from a given triangle ABC by a right line DF drawn from a given point D without that triangle, having a given ratio to the whole triangle, the thing may be done thus. Continue out the side AC to meet GD parallel to AB in the point G ; and divide the side AG so in E , that AE be to AC in the given ratio. Then make [by 12. 6.] as GD is to AB , so is AE to AH . And again make as GA is to AF , so is HF to AH ; then if the right line DF be drawn, it will cut off the triangle AIF from the given triangle ABC having the same ratio to the triangle ABC , as the triangle ABE has; that is, the given ratio.

PROP.

PROP. VIII. THEOR.

If there be two trapeziums $ABCD$, $EFGH$ such that their diagonals AC , BD , and EG , FH be reciprocally proportional; that is, the rectangle under AC and BD equal to the rectangle under EG , FH , and the angles AQB , EPF ; BQC , FPG , made by the diagonals, be equal, those trapeziums will be equal to one another.



For from B and D , F and H , draw the perpendiculars BK , DI ; and FM , HL upon the diagonals AC , EG .

Then because of the equiangular right angled triangle BQK , DIQ , it will be as BK is to DI , so is BQ to DQ ; and [by 18. 5.] as BK and DI is to BQ and DQ , so is BK to BQ : Again because of the similar triangles FPM , HPL ; it will be as FM to LH , so is FP to PH : and [by 18. 5.] as FM and LH is to FP and PH , so is FM to FP . But FM is to FP , as BK is

to BQ ; because the right angled triangles FPM , BQK are similar. Therefore BK and DI will be to BQ and DQ , as FM and LH is to FP and PH ; and [by 1. 6.] as the rectangle under BK and DI together and AC , is to the rectangle under AC and DB , so will the rectangle under FM and LH together and EG , be to the rectangle under FH and EG ; and as the rectangle under BK and DI together and AC , is to the rectangle under FM and HL together, and EG , so will the rectangle under AC and DB , be to the rectangle under EG and FH ; and so will half the first and second rectangles be to the rectangles under AC and

and DB , and EG, FH ; that is, the trapezium $ABCD$ will be to the trapezium $EFGH$, as the rectangle under BD and AC is to the rectangle under FH and EG . Therefore since [by supposition] the rectangle under AC, DB is equal to the rectangle under EG, FH , the trapeziums $ABCD, EFGH$ will be equal to one another.

Hence these sort of trapeziums may be said to be reciprocal figures, as well as those triangles and parallelograms which are called such by Euclid.

PROP. IX. THEOR.

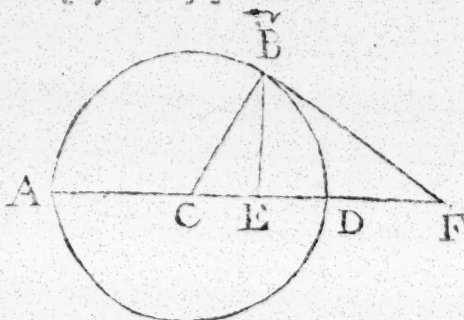
If any tangent to a circle intersects a semidiameter continued out, and from the point of contact be drawn a perpendicular upon that semidiameter, the distance from the centre of the circle to that perpendicular, the semidiameter of the circle, and the distance from the centre to the intersection of the tangent with the continuation of the semidiameter, will be proportionals.

Let ABD be a circle whose centre is C , and CD a semidiameter thereof; and let a tangent BF touching this circle in the point B meet the semidiameter CD continued out in the point F ; also let the right line BE drawn from the point B of contact perpendicular to the semidiameter CD meet the same in the point E . I say the right lines CE, CD, CF will be proportionals.

For draw the semidiameter CB .

Then because [by construction] the angle BEC is a right angle, as also the angle CBF [by 18. 3]. And since the angle BCE is common to both the right angled triangles CBE, CBF . Therefore [by 32. 1.] the triangles CBE, CBF will be equiangular. Wherefore [by 4. 6.] as CE is to CB , so will CB be to CF . But CD is equal to CB . Therefore as CE is to CD , so is CD to CF .

Wherefore, &c. Which was to be demonstrated.



PROP.

PROP. X. THEOR.

If a right line cutting a circle makes an angle at one end of the common base of two segments of that circle equal to the angle in either segment. and any other right line be drawn from the other end of that base to the first mentioned right line, the part of this last line within the circle, the common base of the segments, and the line itself, will be proportionals.

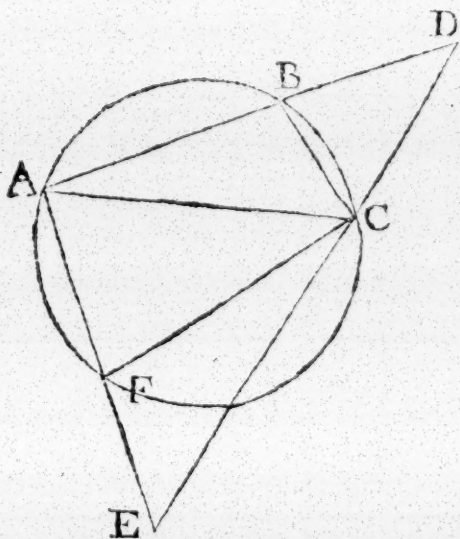
Let the right line AC be the common base of the segments ABC , AFC of a circle, and let the right line ED cutting the circle, make the angle ACD equal to the angle ABC in the segment ABC , or the angle ACE equal to the angle AFC in the segment AFC , and let any right line AD , cutting the circle in B , or any right line AE cutting the circle in F , be drawn from the end A of the common base AC of the segments, to the line ED : I say the right lines AB , AC , AD , as also AF , AC , AE will be proportionals.

For join CB , CF .

Then because [by construction] the angle ACD is equal to the angle ABC , and since [by 21. 3.] the angle ABC is always of the same magnitude however the line AD be drawn. And because the angle A is common to both the triangles ABC , ACD ; therefore [by 32. 1.] the angles CDB , ACB will be equal; and so the triangles ABC , ACD will be always equiangular. Therefore [by 4. 6.] as AB is to AC , so will AC be to AD . In like manner the triangles ACE , ACF will be always equiangular. Wherefore [by 4. 6.] as AF is to AC , so will AC be to AE .

Therefore, &c. Which was to be demonstrated.

Corollary



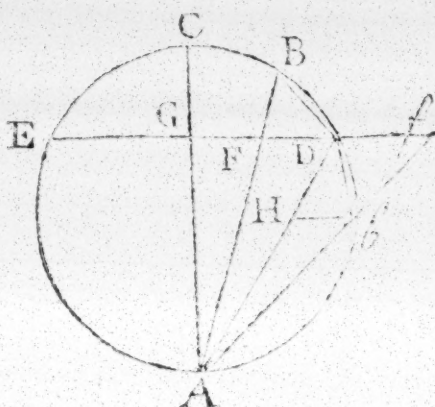
Corollary: Hence if any right line AD , drawn from A , cuts the circle in B , the rectangle under AD and AB will be always of the same magnitude, viz. equal to the square of AC .

PROP. XI.

If AC be the diameter of a circle, which the right line DE cuts at right angles; and from the end A of the diameter the right line AB be drawn meeting the circumference in B , and the right line DE in F . I say the three right lines AB , AD , AF will be proportionals.

For first let the intersection F fall within the circle, and draw BD .

Then because the arches AE , AD are equal, the angles at the circumference EDA , ABD standing upon them will be equal. And so in the triangles ABD , ADF the angles ABD , ADF [by



21. 3.] are equal. But the angle at A is common to both triangles. Wherefore [by 32. 1.] the remaining angles ADB , AFD will be equal to one another. And so the triangles ABD , ADF will be equiangular. Wherefore [by 4. 6.] as BA is to AD , so will AD be to AF . But now let the point of intersection fall without the circle, and draw bh parallel to DE , meeting the right line AD in H . Then from what has been already demonstrated, it will be as DA is to Ab , so is Ab to AH , the triangles ABD , AHb being equiangular; that is [by construction and 2. 6.] so will Af be to AD , therefore Af , AD , Ab will again be proportionals.

Therefore, &c. Which was to be demonstrated.

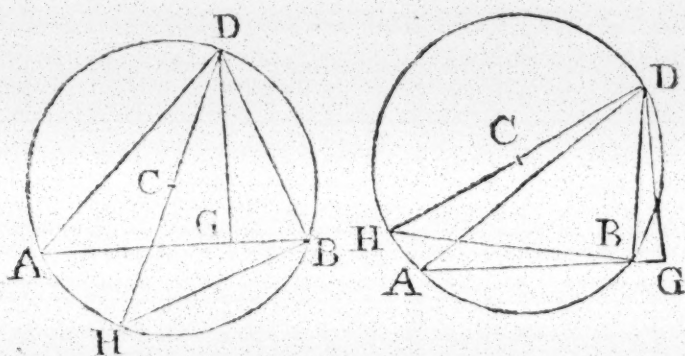
PROP.

PROP. XII. THEOR.

If any triangle be inscribed in a circle, and a perpendicular be let fall from the vertical angle upon the base, the rectangle under the sides of the triangle will be equal to the rectangle under that perpendicular and the diameter of the circumscribed circle.

Let ADB be a triangle, whose altitude is DG , and let a circle, whose centre is C , circumscribe it. Draw the diameter DH , and join HB : I say the rectangle under the sides AD, DB will be equal to the rectangle under the perpendicular DG , and the diameter DH of the circumscribed circle.

For because the angles A, H [by 21. 3.] are equal, as standing upon the same right line DB , and the angles



G, DBH right angles [by supposition, and 31. 3.] the triangles ADG, DBH will be similar. And therefore as AD is to DG , so will DH be to DB . Therefore [by 16. 6.] the rectangle under the sides AD, DB of the inscribed triangle will be equal to the rectangle under the perpendicular DG and the diameter DH of the circumscribing circle.

Therefore, &c. Which was to be demonstrated.

PROP. XIII. THEOR.

If any trapezium be inscribed in a circle, the rectangle under the diagonals will be equal to the sum of the rectangles under each of the opposite sides.

Let there be a trapezium $ABCD$ inscribed in a circle, whose diagonals are AC, BD . I say the rectangle under

AC, BD

AC , BD will be equal to the sum of the rectangles under the opposite sides, viz. to the rectangle under AB and DC , together with the rectangle under AD and BC .

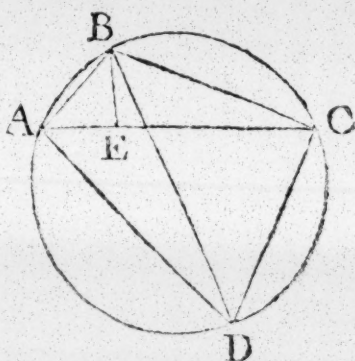
For make the angle ABE equal to the angle CBD . Then the triangles ABD , BCE will be similar, because the angles BDA , BCA being both in the same segment, are [by 21. 3.] equal to one another, and the angle ABD is equal to the angle EBC , because of the addition of the common angle EBD to the equal ones ABE , DBC . Therefore as BC is to CE , so will BD be to DA ; and so [by 16. 6.] the rectangle under BC and DA , will be equal to the rectangle under CE and BD . Also the triangles ABE , CBD will be similar, because the angles BAE , BDC in the same segment are [by 21. 3.] equal to one another. Therefore as AE is to AB , so will CD be to BD . Consequently the rectangle under AB and CD will [by 16. 6.] be equal to the rectangle under AE and BD . But the rectangle under BD and AC [by 1. 2.] is equal to the rectangle under AE and BD , and that under EC and BD . Therefore since it has been proved, that the rectangle under CE and BD is equal to the rectangle under BC and DA , and the rectangle under AE and BD equal to the rectangle under AB and CD ; the rectangle under AC and BD will be equal to the sum of the rectangles under BC and AD , and AB and DC .

Therefore, &c. Which was to be demonstrated.

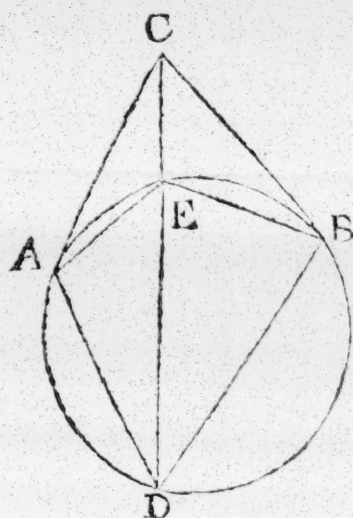
This is a most elegant and useful proposition, being the fountain and foundation of many fine theorems and useful effections, relating to the properties of right lines drawn in a circle. The theorem is Ptolemy's, and is to be found in his *Almagest*.

PROP. XIV. THEOR.

If two tangents CA , CB to a circle be drawn from any point C , and from C any right line CD be drawn cutting the circumference of the circle in the points E , D and the trapezium $AEBD$ be completed



pleated within the circle; the rectangles under the opposite sides AE and DB , and AD and EB will be equal to one another.

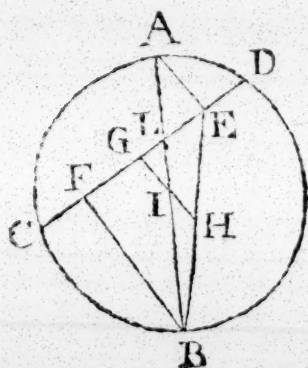


For because [by 32. 3.] the angles CAE , CBE made by the right lines AE , EB are equal to the angles ADE , EDB in the alternate segments of the circle, the triangles ACE , DBC ; and BCE , BCD will be similar: therefore it will be [by 4. 6.] as AE is to AC , so is AD to DC ; and as DB to DC , so is BE to CB . Therefore [by prop. 1. of the additions to the sixth book] the four rectangles under AE and DB ; AC and DC ; AD and BE ; and DC and CB , will be proportional. But because the tangents AC , BC are equal, and so the rectangle under AC and DC equal to the rectangle under CB and DC . Therefore [by 9. 5.] the rectangle under AE and DB will be equal to the rectangle under AD and BE .

Therefore, &c. Which was to be demonstrated.

PROP. XV. THEOR.

In a circle if AB be the diameter, and any right line CD cuts the same in the point L , and from the extremes A and B of that diameter two perpendiculars AE , BF be drawn to that right line CD ; the segments CF , ED of it will be equal to one another.



For join the points E , B , and from the centre I draw IG perpendicular to CD , and continue it to H in the line EB .

Then [by 3. 3.] will CG be equal to GD . And because IG , AE are parallel, and since BI is equal to IA ; BH will be [by 4. 6.] equal to HE , and FG to GE . Therefore the difference between CG and FG will

will be equal to the difference between GD and GE , that is, CF will be equal to ED .

Therefore, &c. Which was to be demonstrated.

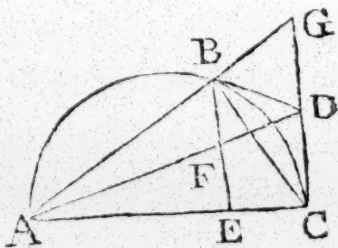
Corollary. Hence if there be any trapezium whose two opposite angles are right angles, and a diagonal be drawn joining these right angles; and if from each of the other angles be drawn a perpendicular upon that diagonal; the segments of this diagonal made by those perpendiculars will be equal to one another.

PROP. XVI.

If CD, BD be two tangents to a circle at B and C , whereof C is the extreme of a diameter AC , and the perpendicular BE be drawn upon AC , and if the points A and D be joined by a right line cutting the perpendicular BE in the point F ; this perpendicular BE will be bisected in the point F .

For thro' B draw AB to meet the tangent CD (continued out) in the point G ; and join BC .

Then because the tangents BD, DC are equal, the angles DCB, DBC [by 5. 1.] will be equal, and the angles DBC, DBG are equal to a right angle, because the angle CBG , which is equal to the angle ABC , is [by 31. 3.] a right angle. Also the angles DCB, CGB [by 32. 1.] are equal to a right angle. Wherefore the angle DBG will be equal to the angle DGB . And so [by 6. 1.] BD , that is CD , will be equal to DG . Wherefore since BE, GC are parallel, EF will be equal to FB .



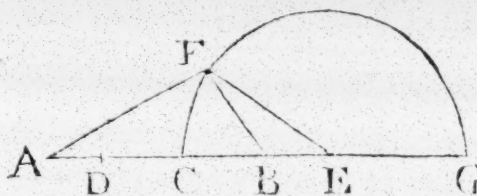
Therefore, &c. Which was to be demonstrated.

Corollary. Hence the perimeter or circuit of any regular polygon circumscribing a circle, is to the perimeter of a regular polygon of half the number of sides inscribed in that circle, as the diameter CA is to the segment AE thereof.

PROP. XVII. THEOR.

If the point B be taken in the diameter CG of a circle, whose centre is E , and the point A in the continuation of that diameter such, that the difference between AC and CB be to CB , as CB is to BE : two right lines AF , BF drawn from those assumed points A and B to any point F in the circumference of the circle, will be in a constant ratio, viz. as AC is to CB .

For make AD equal to the difference between AC and CB , and draw the semidiameter EF .



Then because [by construction] AD is to DC , as CB is to BE ; that is, [by compounding by the 18th 5.] AC is to CD , or CB , as CE is to EB :

it will be [by 12. 5.] as AE is to CE , that is, EF , so is CE or EF to EB . Therefore because the triangles AEF , FEB have one angle E common to both, and the sides about it proportional: also the remaining sides AF , FB [by 6. 6.] will be to one another as AE is to EF , or EC , that is, CE to EB . But as CE is to EB , so is AC to CB . Wherefore it will be as AF is to FB , so is AC to CB .

Therefore, &c. Which was to be demonstrated.

This is a famous and useful proposition of Apollonius's.

SCHOLIUM.

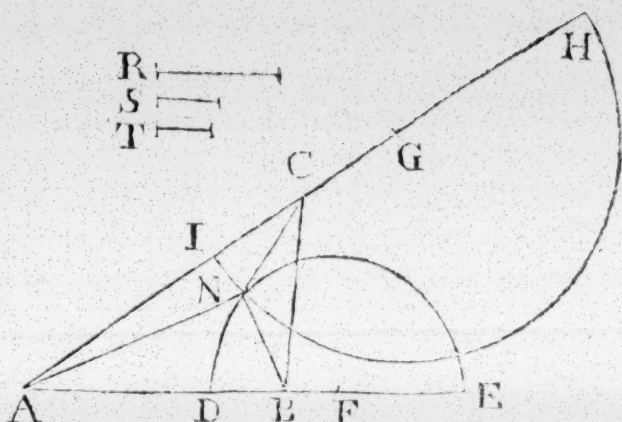
By means of this proposition it is easy to find a point N within a triangle ABC such that drawing to it three right lines from the three angles of that triangle they shall be in given ratios, viz. respectively as the lines R , S , T .

For divide the side AB of the given triangle in the point D so, that AD be to DB , as R is to S , (which may be easily done by putting the lines R , S in the same direction, and then [by 10. 6.] dividing the line AB in the same manner as the sum of the lines R and S is divided.) This done, find [by 12. 6.] a fourth proportional right line BE to AB , DB and the difference between AD and DB . Bisect DE in F , and about F with the distance DF or FE describe a circle.

Again, divide the side AC of the given triangle so in I , that

AI

AI be to *IC*, as *s* is to *T*; and make *CH* a fourth proportional to *AC*, *IC*, and the difference between *AI* and *IC*. Bisect *IH* in *G*, and with the distance *IG*, or *GH*, describe a



circle cutting the former one in the point *N*. Draw the right lines *AN*, *BN*, *CN*, and these will be to one another in the given ratios of *R*, *S*, *T*.

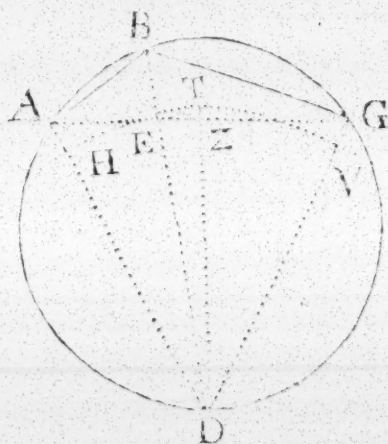
Or the centre *F* may be found by taking *BF* a third proportional to *DB* and the difference between *AD* and *DB*. And so may the centre *G* be found, by taking *CG* equal to a third proportional between *IC* and the difference between *AI* and *IC*.

PROP. XVIII. THEOR.

In a circle the ratio of a greater arch BG to a lesser AB, will be greater than that of the right line BG joining the ends of the greater arch, to that of the right line AB joining the ends of the lesser arch AB.

For draw the right line *AG*, and the equal right lines *AD*, *DG*, and join the points *D*, *B* by a right line cutting *AG* in *E*, and draw the perpendicular *DZ* to *AG*, and having described the arch *HTV* about *D*, as a centre, with the distance *DE*, continue out *DZ* to meet the same in *T*.

Now because the line *DB* bisects the angle *ABG*, for the lines *AD*, *DG* being equal [by construction] the arches *AD*, *GD*, and so the angles *ABD*, *DBG* [by 29. and 21. 3.] will be equal. Therefore [by 3. 6.] as *BG* is to *AB*, so will *EG* be to *EA*. And since *BG* is greater than *AB*; *EG* will be greater than *AE*. Where-



fore the point z bisecting AG will fall between E and G . And because the side DE of the right angled triangle DEZ is greater than DZ , and the side DA of the right angled triangle AZD is greater than DE . Therefore the arch HE will cut the circumference of the circle below A , and fall at T above the point z . Again, since the sector EDT is greater than the triangle EDZ : the ratio of the sector EDT to the sector EDH , will be greater than the ratio of the triangle EDZ to the sector EDH . But the ratio of the triangle EDZ to the triangle EDA is less than the ratio of the triangle EDZ to the sector EDH . Therefore much more will the ratio of the sector EDT to the sector EDH be greater than that of the triangle EDZ to the triangle EDA . But [by 33. 6.] in the same circle as sector is to sector, so is the arch of the one to the arch of the other; and as arch is to arch, so is one angle at the centre to another: also [by 1. 6.] as the triangle EDZ is to the triangle EDA , so is ZE to EA . Therefore the ratio of the angle EDV to the angle EDA , will be greater than the ratio of GE to EA . But the angle GDB is to the angle BDA [by 33. 6.] as the arch BG , is to the arch AB ; and GE is to EA , as BG is to AB . Therefore the ratio of the arch BG to the arch AB is greater than that of the right line BG joining the extremes of one, to the right line AB joining the extremes of the other.

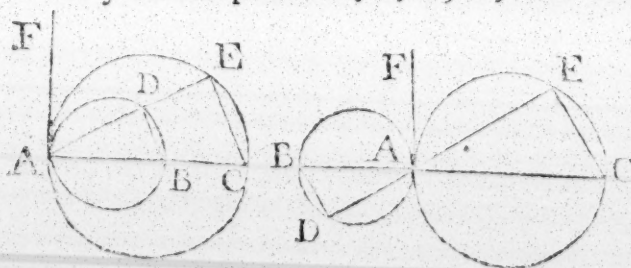
Therefore, &c. Which was to be demonstrated.

This is one of Ptolemy's theorems, with his most ingenious demonstration.

PROP. XIX. THEOR.

If two circles touch one another at A , and thro' A be drawn two right lines cutting the circles in D, E, B, C . I say the right lines AB, AC, AD, AE will be proportional.

For join the points $D, B; E, C$; and conceive AF to be a tangent to the



circles at A . Then [by 32. 3.] the angles at B and C will be equal to the angle

$\angle FAE$: and since, in the first figure, the angle at A is common to both the triangles ADB, AEC , but in the second figure, the vertical angles at A are equal to one another, the remaining angles ADB, AEC [by 32. 1.] will be equal too; and so the triangles ADB, AEC will be equiangular. Wherefore [by 4. 6.] it will be as AB is to AC , so is AD to AE .

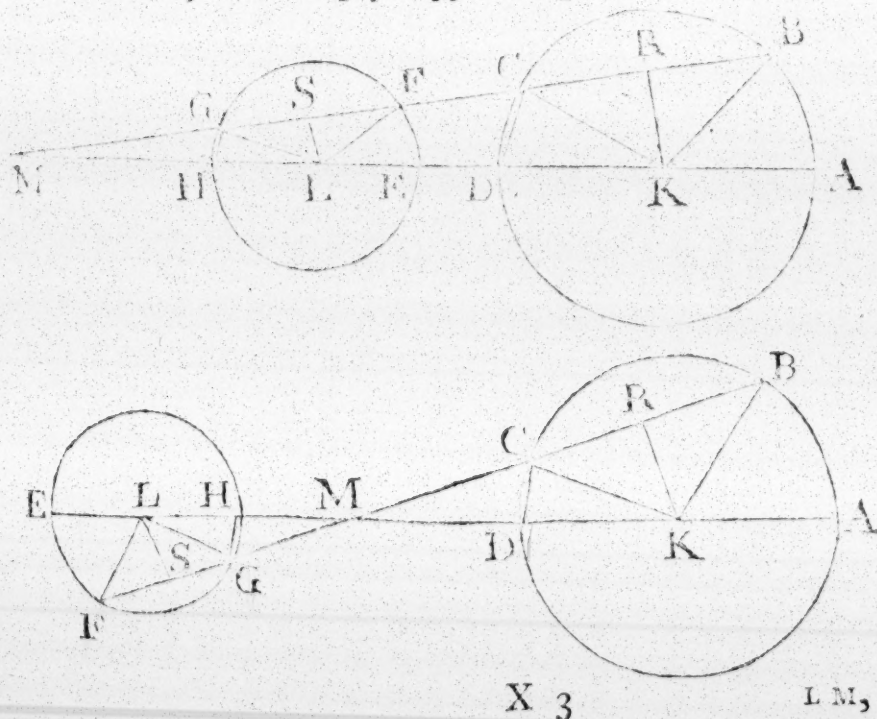
Therefore, &c. Which was to be demonstrated.

PROP. XX. THEOR.

If the right line KL , or its continuation, joining the centres K, L of two circles whose diameters are EH, DA , be so divided in M , that KM be to LM , as AK is to EL . And if from or thro' M be drawn any right line MB cutting the circles in the points G, F, C, B , I say first the segments of the circles CB, GF will be similar. Secondly the rectangle under MG and MB will be equal to the rectangle under MH and MA . Thirdly the rectangle under ME and MD equal to the rectangle under MF and MC .

Draw the semidiameters GL, LF, CK, KB ; and join the points $G, H; C, D$.

Then, first, because [by supposition] it is as KM is to



LM , so is KB to FL ; the semidiameters BK, FL will [by 2. 6.] be parallel. And therefore [by 29. 1.] the angle KBC will be equal to the angle $LF G$.

In like manner, because [by supposition] as KM is to LM , so is KC to LG ; the semidiameters KC, LG will be parallel; and so the angle KCB equal to the angle LGF . And therefore the remaining angles BKC, FLG will be equal to one another: that is, the arch BC [by 33. 6.] is the same part of the circumference of the circle ABC , as the arch GF is of the circumference of the circle $F GH$. And so BC, FG will be similar segments of those circles.

Secondly, From the centres K, L draw the perpendiculars KR, LS to MB .

Then because BC, FG are similar segments, and RC, SG are half of the right lines CB, GF [by 3. 3.]. Therefore the triangles KRC, LSG will be similar; and so KC, LG will be parallel; and the angles CKD, GLH equal, and CD, GH parallel. Wherefore as MD is to MC , so is MH to MG . But as MD is to MC , so [by supposition] is MB to MA . Therefore [by 11. 5.] as MB is to MA , so will MH be to MG . Wherefore [by 16. 6.] the rectangle under MB and MG will be equal to the rectangle under MA and MH .

Thirdly, because the four points E, F, G, H are in a circle, it will be [by 36. 3. and 16. 6.] as MH is to MG , so is MF to ME . But MH is to MG , as MD is to MC . Therefore [by 11. 5.] as MF is to ME , so will MD be to MC ; and so [by 16. 6.] the rectangle under ME and MD will be equal to the rectangle under MF and MC .

Therefore, &c. Which was to be demonstrated.

SCHOLIUM.

This proposition, and the converse of the last, are lemmas of Vieta's, in his Apollonius Gallus. By means of the former he describes a circle through two given points to touch a right line given in position. By means of the other, he describes a circle through a given point to touch two given circles.

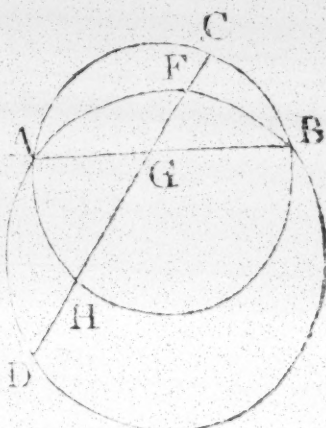
PROP.

PROP. XXI. PROBL.

If two circles intersect one another, and any right line be drawn cutting the circles, this will be proportionally divided by the circumferences of the circles.

Let the circles ACB , AFB intersect one another in the points A and B , and let AB be drawn. Also let any right line DC cut both the circles, viz. ACB in the points C and H , and the circle AFB in the points D and F , and the right line AB in the point G : I say the right line DC is divided proportionally in the points H, G, F , that is, as FC is to GF , so will DH be to HG .

For because in the circle $ACBH$, the lines AG, GC, GH, GB , [by 35. 3. and 16. 6.] are proportional, and the lines AG, GF, DG, GB , in the circle $AFDB$, the lines GC, GF, DG, GH will [by equality] be proportional. And therefore [by 17. 5.] as FC is to GF , so will DH be to HG .



Therefore, &c. Which was to be demonstrated.

E U C L I D's E L E M E N T S. B O O K XI.

D E F I N I T I O N S.

1. **A** Solid is that which has length, breadth, and thickness.
2. The bound of a solid is a superficies^a.
3. A right line is perpendicular to a plane, when it makes right angles with all the right lines that touch it, and are drawn in the said plane.
4. A plane is perpendicular to a plane, when the right lines drawn in one plane perpendicular to the common section of the planes be all at right angles to the other plane.
5. The inclination of a right line to a plane, is, when a perpendicular is drawn from the highest point of that line to the plane, and another line drawn from the point in which the perpendicular cuts the plane, to the end of the said line which is in the same plane, viz. the acute angle contained under the joined and inclining line.
6. The inclination of a plane to a plane, is an acute angle contained under the right lines, which being drawn in either of the planes to the same point of their common section is at right angles to the common section.
7. Planes are said to have similar inclinations, when the aforesaid angles of their inclination are equal to one another.
8. Parallel planes are those which being produced will never meet.

^a It might be better to say, the bound or bounds of a solid, is one or more superficies, for a sphere has but one bound or superficies, and a cube six.

9. Similar solid figures are those which are contained under equal numbers of similar planes.

10. Equal and similar solid figures, are such as are contained under similar planes equal in number and magnitude.

11. A solid angle is the inclination of more than two right lines, which touch one another and be not in the same plane. Or thus, a solid angle is that which is contained under more than two plane-angles not lying in the same plane, and standing at one point^b.

12. A pyramid is a solid figure contained under several planes standing upon a plane, and meeting in one point.

13. A prism is a solid figure contained under planes, the two opposite of which are equal, similar, and parallel, but the others are parallelograms^c.

14. A sphere is a solid figure generated by the entire revolution of a semicircle about its diameter, remaining at rest^d.

15. The axis of a sphere is that unmoveable right line about which the semicircle revolves.

16. The centre of a sphere is the same as that of the semicircle generating it.

17. The diameter of a sphere is a right line drawn thro' the centre, and both ways terminating in the superficies of the sphere.

18. A cone is a solid figure generated by the intire revolution of a right angled triangle about one of the sides containing the right angle, which remains at rest during

^b The former part of this definition, is not so clear and distinct as the latter.

^c A pyramid may perhaps be better defined thus: If a right line always passing thro' a fix'd point over a plane, moves from one angle of any right lined figure, in that plane, along every side of that figure till it returns to the angle from whence it first moved, the solid contained under the superficies described by the moving right line and the figure upon the plane is called a pyramid, and a prism may be defined much after the same way.

^d All definitions of figures from their generations are esteemed better than those derived from the properties of the figures; and therefore this definition of a sphere is better than if it had been called a solid contained under one superficies, having a point within it such, that all right lines drawn from that point to the superficies are equal to one another.

the motion of the triangle. And if the fixed side be equal to the other side containing the right angle, the cone is a right angled cone: If it be less, it is an obtuse angled cone; if greater, an acute angled cone^e.

19. The axis of the cone is that fixed side of the triangle generating it^f.

20. The base of a cone is the circle generated by the motion of one of the sides of the triangle generating the cone.

21. A cylinder is a solid figure made by the intine revolution of a right angled parallelogram about one of its sides, which remains at rest during the revolution of the parallelogram^g.

22. The axis of the cylinder is that side of the revolving parallelogram which remains at rest, while the cylinder is generating^h.

23. The bases of a cylinder are the circles described by two of the opposite sides of the parallelogram generating the cylinder.

24. Similar cones, and cylinders are those whose axes and the diameters of their bases are proportional.

^e This definition of a cone is too scanty, it only taking in a right cone, viz. that whose axis is at right angles to the plane of the base. Nor is the distinction of a right cone, into right angled, oblique angled, and acute angled, of any use, at least in these Elements.

The full definition of a cone is this. If a right line always passing thro' a fixed point over a plane, moves from one point of the circumference of a circle in that plane, quite round the circumference, returning again to the point from whence it first moved, the solid contained under the superficies generated by that moving right line and the circle in the plane, is called a cone.

^f The base of a cone is that circle, and the axis, the right line drawn from the centre of the base to the fixed point or vertex.

^g This definition of a cylinder is only particular, viz. only extending to a right cylinder whose axis is at right angles to the plane of either of its bases. The general definition is this. If a right line moves round every part of the circumferences of two equal and parallel circles, the solid comprehended under the superficies generated by the moving right line, and the two parallel circles, is called a cylinder.

^h The axis of a cylinder is rather the right line drawn thro' the centres of the parallel circles or bases of the cylinder.

25. A cube

25. A cube is a solid figure contained under six equal squares.

26. A tetrahedron is a solid figure contained under four equal equilateral triangles.

27. An octahedron is a solid figure contained under eight equal equilateral triangles.

28. A dodecahedron is a solid figure contained under twelve equal equilateral pentagons.

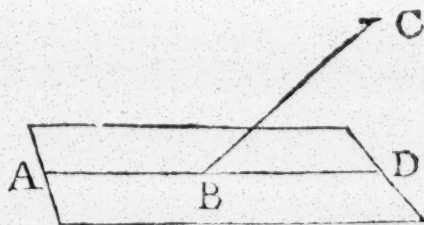
29. An icofahedron is a solid figure contained under twenty equal equilateral triangles¹.

¹ The 27th, 28th, and 29th definitions of an octahedron, dodecahedron, and icofahedron, and indeed the whole theory of these solids, laid down by Euclid in his 13th, 14th, and 15th books seem to be more curious than useful; and I have often wondered why the five Platonic bodies should be held in such esteem by the ancient Platonic philosophers, as we find they were, and that even Euclid, who himself was a sectator of Plato, should have compiled the whole body of his Elements for the sake of this contemplation, as Proclus says it is reported he did.

PROP. I. THEOR.

One part of a right line cannot be in a plane, and the other elevated above it.

For if possible let the part AB of the right line ABC lye in a plane underneath, and the other part BC be elevated above it. Then will some right line being the direct continuation of AB also be in that same plane. Let this be BD . Wherefore two right lines ABC , ABD have both one common segment



AB , which is impossible; for one right line cannot meet another in more points than one; otherwise the right lines will coincide.

Therefore one part of a right line cannot lye in a plane, and the other part be elevated above it. Which was to be demonstrated.

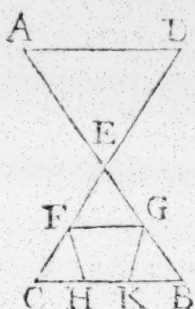
PROP.

PROP. II. THEOR.

If two right lines mutually cut each other, they are both in the same plane; also every triangle lies in one plane^k.

For let two right lines AB, CD cut one another in the point E : I say the right lines AB, CD are both in the same plane, also every triangle lies in one plane.

For take any points F, G in EB, EC . Join CB, FG , and draw FH, GK : First, I say the triangle $EB C$ lies in one plane: For if one part



FHC or GBK of the triangle $EB C$ lies in one plane, and the remaining part lies in another plane; then will one part of the lines EC, EB fall in one plane, and another part in another plane. And if the part $FCBG$ of the triangle ECB be in one plane, and the remaining part in another,

one part of the right lines EC, EB , will be in one plane, and another part in another plane; which we have proved to be absurd. Therefore the triangle $EB C$ lies in one plane: But in that plane wherein the triangle $BC E$ lies, both the right lines EC, EB do lie: And in that plane wherein are the right lines EC, EB , are [by I. 11.] the right lines AB, CD . Therefore the right lines AB, CD are both in the same plane, and every triangle is in the same plane. Which was to be demonstrated.

^k From hence it appears why a stool, table, &c. with only three legs, will always stand firm, when those with four, or more, oftentimes will not.

PROP. III. THEOR.

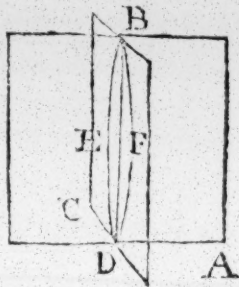
If two planes cut one another, their common section will be a right line.

For let two planes AB, BC cut one another, and let their common section be the line DE : I say DE is a right line.

For if it be not, from the point D to B in the plane AB , draw the right line DEB , and in the plane BC , the right line DFB : Therefore the right lines DEB, DFB will

have

have the same bounds, and include a space. Which [by 12. ax.] is impossible: Wherefore DEB , DFB are not right lines. After the same manner we demonstrate that no other line except BD , the common section of the two planes AB , CD drawn from the point D to B , can be a right line.



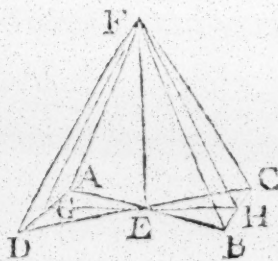
Therefore if two planes cut one another, their common section will be a right line. Which was to be demonstrated.

PROP. IV. THEOR.

If a right line be at right angles to two right lines cutting one another in the common section [of two planes], it will be at right angles to the plane drawn thro' those lines.

For let any right line EF be at right angles to the right lines AB , CD , cutting one another in the point E : I say also that EF is at right angles to the plane drawn through AB , CD .

For take the equal right lines AE , EB , CE , ED ; and through the point E draw the right line GEH any how. Join AD , CB ; and from any point F draw FA , FG , FD , FC , FH , FB . Then because the two right lines AE , ED are equal to the two right lines CE , EB , and [by 15. 1.]



they contain equal angles, the base AD [by 4. 1.] will be equal to the base CB , and the triangle AED equal to the triangle CEB , and also the angle DAE equal to the angle ECB ; and [by 15. 1.] the angle AEG is equal to the angle BEH : Therefore the two triangles AGE , EBH have two angles of the one equal to two angles of the other, each to each, and one side AE of the one equal to one side EB of the other, which lies between the equal angles: Wherefore [by 26. 1.] the remaining sides of the one are equal to the remaining sides of the other: Wherefore GE is equal to EH , and AG to BH . And because AE is equal to EB , and FE is common, and at right angles; the base FA [by 4. 1.] will be equal to the base FB .

By

By the same reason, FC will also be equal to FD . Again, because AD is equal to CB , and FA to FB ; the two sides FA, AD will be equal to the two sides FB, BC , each to each; and the base FD has been proved to be equal to the base FC : Therefore the angle FAD [by 8. 1.] is equal to the angle FBC . Again, AG has been proved to be equal to BH : But FA also is equal to FB : Therefore the two sides FA, AG are equal to the two sides FB, BH . And the angle FAG has been proved to be equal to the angle $F BH$: Therefore the base FG [by 4. 1.] is equal to the base FH . And again, because GE has been proved to be equal to EH , and EF is common; the two sides GE, EF , will be equal to the two sides HE, EF , and the base FH is equal to the base FG : Therefore the angle GEF is equal to the angle HEF [by 8. 1.], and so the angles GEF, HEF are each a right angle: Therefore FE is at right angles to GH any how drawn thro' E . After the same manner we demonstrate that FE is at right angles to all right lines that touch it, and are in the same plane. But [by 3. def. 11.] one plane is perpendicular to another, when it is at right angles to all the right lines touching it, and being in the same plane. Wherefore EF will be at right angles to the plane drawn through the right lines AB, CD .

If therefore a right line be at right angles to two right lines cutting one another in the common section [of two planes;] it will be at right angles to the plane drawn thro' those lines. Which was to be demonstrated.

PROP. V. THEOR.

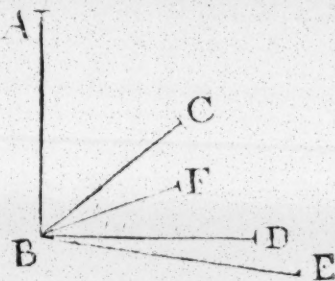
If a right line stands at right angles to three right lines meeting one another in their common section; those three right lines will be all in the same plane.

For let any right line AB stand at right angles to the three right lines BC, BD, BE , in their point of contact B : I say the right lines BC, BD, BE are all three in the same plane.

For if not, let, if possible, BD, BE be in one plane, and BC above it; and let the plane passing through AB, BC be produced: Then the common section will make

[by

[by 3. 11.] a right line in the plane beneath, which let be BF . Therefore the three right lines AB , BC , BF are in one plane drawn thro' AB , BC . And because AB stands at right angles to BD , BE , it will be at right angles to the plane drawn thro' BD , BE [by 4. 11.]. But the plane drawn thro' DB , BE is the plane beneath: Therefore AB is perpendicular to the plane beneath: Wherefore [by 3. def. 11.] it will be perpendicular to all right lines touching it, which are in the same plane: But BF being in the plane beneath, touches it: Therefore ABF is a right angle. But the angle ABC is also supposed to be a right angle: Therefore the angle ABF is equal to the angle ABC ; and they are both in the same plane; which [by 9. ax.] is impossible. Therefore the right line BC is not above the plane beneath: Therefore the three right lines BC , BD , BE are all in one plane.



Wherefore if a right line stands at right angles to three right lines meeting one another in their common section; those three right lines will be all in the same plane. Which was to be demonstrated.

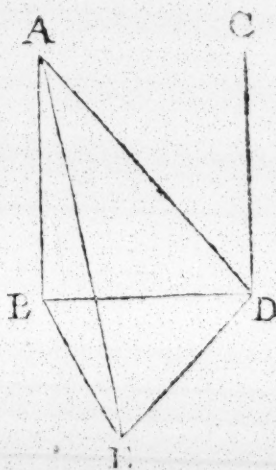
PROP. VI. THEOR.

If two right lines be at right angles to the same plane, those right lines will be parallel to one another.

For let two right lines AB , CD be at right angles to a plane: I say AB is parallel to CD .

For let them meet the plane in the points B , D ; and join BD by a right line, and draw DE in the plane at right angles to BD ; make DE equal to AB , and join BE , AE , AD .

Then because AB is perpendicular to the plane; it will be [by 3. def. 11.] perpendicular to all right lines in that plane which touch it. But BD , BE , being both in the plane, touch AB :



Therefore

Therefore the angles ABD , ABE are each of them right angles. Also by the same reason the angles CDB , CDE are each right angles: And because AB is equal to DE , and BD is common, the two sides AB , BD are equal to the two sides ED , DB , and they contain a right angle: Therefore [by 4. 1.] the base AD is equal to the base BE . And because AB is equal to DE , and AD to BE , the two sides AB , BE are equal to the two sides ED , DB , and their base AE is common: Therefore [by 8. 1.] the angle ABE is equal to EDA . But ABE is a right angle. Therefore also EDA is a right angle: and so ED is perpendicular to DA . But it is also perpendicular to BD , DC : Wherefore ED stands in the point of contact at right angles to the three right lines BD , DA , DC : Wherefore the three right lines BD , DA , DC [by 5. 11.] are in one plane. But AB is in the same plane wherein are BD , DA , for [by 2. 11.] every triangle is in one plane: Therefore AB , BD , DC are in one plane; and ABD , BDC are each right angles: Wherefore [by 28. 1.] AB is parallel to CD .

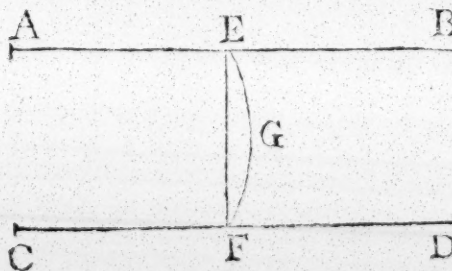
Therefore if two right lines be perpendicular to the same plane; those right lines will be parallel. Which was to be demonstrated.

PROP. VII. THEOR.

If two right lines be parallel, and any points be taken in each of them; the right line which joins those points will be in the same plane wherein are the parallels.

Let there be two parallel right lines AB , CD , and let any points E , F be taken in each of them: I say the right line joining the points E , F are in the same plane wherein are the parallels.

For if it be not, let it, if possible, be elevated above the same plane, as EGF ; and thro' EGF draw a plane, which [by 3. 11.] makes a section with the plane wherein are the parallels, being a right line, as EF . Then the two right lines GF , EF will include a space,



space, which [by 12. ax.] is impossible: Therefore the right line drawn from the point E to F cannot be elevated above the plane passing thro' the parallels AB, CD , consequently it is in the same.

Wherefore if two right lines be parallel, and any points be taken in them, the right line joining the said points will be in the same plane wherein are the parallels. Which was to be demonstrated.

PROP. VIII. THEOR.

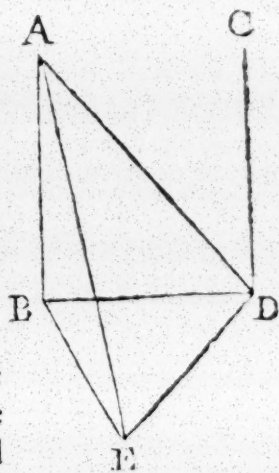
If there be two right lines parallel, and one of them be at right angles to some plane, the other will also be at right angles to the same plane.

Let there be two parallel right lines AB, CD , and let one of them, as AB , be at right angles to some given plane; the other CD will also be at right angles to that plane.

For let AB, CD meet the plane in the points B, D ; and join BD : Then [by 7. 11.] AB, DC, DB are all in one plane. Draw DE in the given plane at right angles to BD , make DE equal to AB , and join BE, AE, AD : Then because AB is perpendicular to the given plane, it will be [by 3. def. 11.] perpendicular to all right lines which touch it, and are in the given plane: Therefore ABD, ABE are each of them right angles. But because the right line BD falls upon the two parallel right lines AB, CD ; the angles ABD, CDB [by 29. 1.] will be equal to two right angles: But ABD is a right angle:

Therefore CDB is also a right angle; and so CD is perpendicular to BD . And because AB is equal to DE , and BD is common, the two sides AB, BD are equal to the two sides ED, DB , and the angle ABD is equal to the angle EDB , for each of them is a right angle; therefore the base AD [by 4. 1.] is equal to the base BE : And because AB is equal to DE , and BE to AD , the two sides AB, BE will be equal to the two sides ED, DA , each to each, and their base AE is common: Wherefore

[by 8. 1.] the angle ABE is equal to the angle EDA : but ABE is a right angle; therefore EDA also is a right angle;



Y

and

and so ED is perpendicular to DA . But it is also perpendicular to DB : Therefore [by 4. 11.] ED will be perpendicular to the plane drawn thro' BD, DA ; and will be at right angles to all right lines which touch it, and are in that same plane. But DC is in the plane drawn thro' BA, AD , because AB, BD [by 2. 11.] are in the plane drawn thro' BD, DA : But [by 7. 11.] AB, BD are in the same plane as DC is: Therefore ED is at right angles to DC ; and so CD is at right angles to DE . But so also is CD to BD . Therefore CD stands at D , at right angles to two right lines DE, DB mutually cutting one another in the common section of the two planes; and accordingly [by 4. 11.] CD is at right angles to the plane drawn thro' DE, DB . But the plane drawn thro' DE, DB is the given plane: Therefore CD will be at right angles to the given plane. Which was to be demonstrated.

PROP. IX. THEOR.

Those right lines which are parallel to the same right line, but not in the same plane with it, are also parallel to one another.

For let AB, CD be each of them parallel to the right line EF , but not lying in the same plane with it: I say AB is parallel to CD .

For take any point G in EF , and from the same draw GH in the plane passing thro' EF, AB , at right angles to EF ; and again draw GK at right angles to EF in the plane passing thro' FE, CD . Then because EF is perpendicular to GH , and to GK ; EF [by 4. 11.] will also be at right angles to the plane passing thro' GH, GK , and EF is parallel to AB :

Therefore [by 8. 11.] AB is also at right angles to the plane passing thro' H, G, K . By the same reason CD is also at right angles to the plane passing thro' H, G, K : Therefore AB, CD , each of them will be at right angles to the plane passing thro' H, G, K . But if two right lines be at right angles to the same plane; [by 6. 11.] they will be parallel to one another: Therefore AB is parallel to CD . Which was to be demonstrated.

PROP.

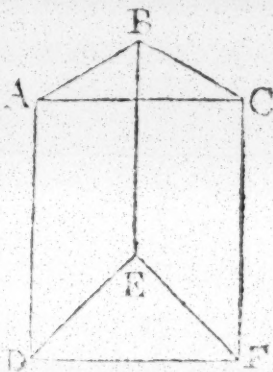
PROP. X. THEOR.

If two right lines touching one another, be parallel to two right lines touching one another, but not being in the same plane; those right lines will contain equal angles.

For let two right lines AB, BC touching one another, be parallel to two right lines DE, EF touching one another, but not in the same plane: I say the angle ABC is equal to the angle DEF .

For take BA, BC, ED, EF equal to one another; and join AD, CF, BE, AC, DF . Then because BA is equal and parallel to ED ; [by 35. I.] AD will be equal and parallel to BE . By the same reason CF will be equal and parallel to BE . Therefore AD, CF are each of them equal and parallel to BE .

But right lines which are parallel to the same right line, but not in the same plane [by 9. 11.] will be parallel to one another: Therefore AD is parallel and equal to CF ; and the right lines AC, DF join them: Wherefore AC is equal and parallel to DF . And because the two sides AB, BC are equal to the two sides DE, EF , and the base AC is equal to the base DF ; the angle ABC [by 8. 1.] will be equal to the angle DEF .



Therefore if two right lines touching one another be parallel to two right lines touching one another, but not being in the same plane, those right lines will contain equal angles. Which was to be demonstrated.

PROP. XI. PROBL.

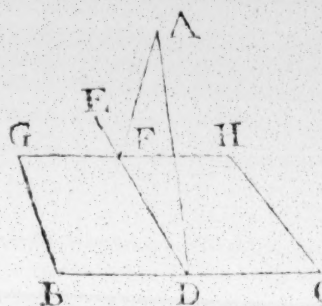
From a given point above a given plane, to draw a right line perpendicular to that plane¹.

Let A be a given point above a given plane: it is required to draw a right line from the point A , perpendicular to that given plane.

Draw any how in the given plane, the right line BC , and [by 12. 1.] from A draw AD perpendicular to BC .

Y 2

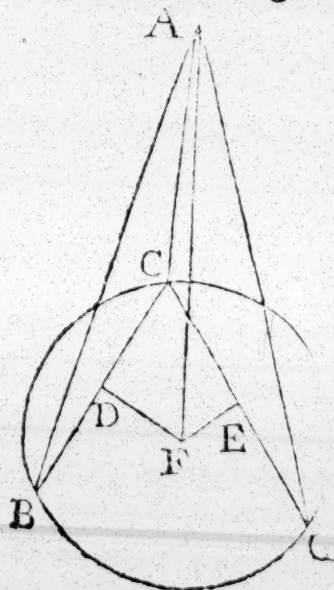
If



If therefore AD be also perpendicular to the given plane; what was required will then be done: But if it be not, [by 11. 1.] draw DE in the given plane from the point D , perpendicular to BC ; and [by 12. 1.] draw AF from the point A , perpendicular to DE ; and thro' F draw GH parallel to BC .

Then because BC is at right angles to AD , DE ; [by 4. 11.] BC will be at right angles to the plane passing thro' ED , DA , and GH is parallel to it. But if there be two parallel right lines, one of which is at right angles to some plane, the other [by 8. 11.] will also be at right angles to that plane: Wherefore GH is at right angles to the plane passing thro' ED , DA ; and so [by 3. def. 11.] it is perpendicular to all right lines touching it, which are in the same plane. But the right line AF touches GH being in the plane passing thro' ED , DA : Therefore GH is perpendicular to FA : Consequently FA is perpendicular to GH . But [by construction] AF also is perpendicular to DE : Therefore AF is perpendicular to GH , and to DE . But if a right line stands at right angles to two right lines touching one another in the common section; [by 4. 11.] it will be at right angles to the plane drawn thro' them: Wherefore FA is at right angles to the plane drawn thro' ED , GH . But the plane drawn thro' ED , GH is the given plane: Therefore AF is perpendicular to this plane.

Wherefore a right line AF is drawn from a given point A above a given plane, perpendicular to that plane. Which was to be done.



¹ The practice of this problem may be thus. From the given point A , draw three right lines AB , AC , AG , of the same length, meeting the plane in the points B , C , G , which may be easily done, with a string or a pair of compasses. Join the points B , C ; C , G ; and having bisected BC , CG in D , E draw the perpendiculars DF , EF to BC , CG , meeting in the point F : Then will the right line AF be perpendicular to the plane BG . The demonstration being easy, I shall omit.

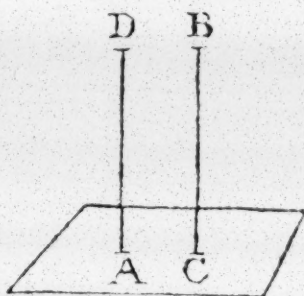
P R O P.

PROP. XII. PROBL.

A plane being given, and a point given in it; to erect a right line from that point at right angles to that plane^m.

Let A be the given point in a given plane: It is required to erect a right line from that point at right angles to that plane.

Conceive some point B to be above the plane, and [by 11. 11.] draw from it the perpendicular BC to the given plane: And [by 31. 1.] thro' A draw AD perpendicular to BC.



Then because the two right lines AD, CB are parallel, and BC one of them is at right angles to the given plane, the other AD [by 8. 11.] will be also at right angles to the given plane.

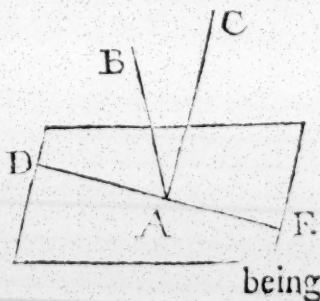
Therefore a plane being given, and a point given in it, a right line is erected from that point at right angles to the plane. Which was to be done.

^m This problem is easily resolved by help of a square, or a parallel ruler.

PROP. XIII. THEOR.

At a point given in a given plane, two right lines cannot be erected perpendicular to the plane, on the same sideⁿ.

For if it be possible, at the point A in the given plane, let two right lines AB, AC be drawn at right angles on the same side; and thro' AB, AC draw a plane, which [by 3. 11.] will make a right line in the given plane drawn thro' A: which let be DAE: then the right lines AB, AC, DAE, are in one plane. But because CA is at right angles to the given plane, [by 3. def. 11.] it will be at right angles to all right lines in that plane which touch it. But the right line DAE,



being in the given plane, touches CA : Therefore CAE is a right angle: by the same reason BAE is also a right angle: Wherefore the angle CAE is equal to BAE , and they are both in one plane. Which [by 9. ax.] is impossible.

Therefore at a given point in a given plane, two right lines cannot be erected at right angles on the same side. Which was to be demonstrated.

" Tho' two right lines cannot be drawn from the same point in a plane perpendicular to that plane on the same side of it, yet two such perpendiculars may be drawn, the one on one side, and the other on the other side of that plane.

PROP. XIV. THEOR.

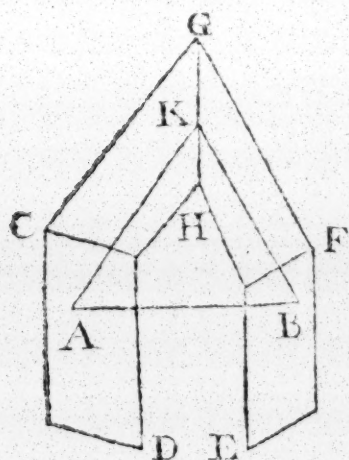
Those planes to which the same right line is perpendicular, are parallel to one another.

For let the right line AB be perpendicular to each of the planes CD, EF : I say those planes are parallel.

For if they be not parallel, they will meet when produced. Let them be produced, and [by 3. 11.] their common section is a right line, which let be GH . Take any point K in GH , and join AK, BK . Then because AB is perpendicular to the plane EF ; [by 3. def. 11.] it will also be perpendicular to the right line BK which is in the plane EF produced: Wherefore ABK is a right angle. By the same reason BAK is also a right angle; and so the two angles ABK, BAK of the triangle ABK , are equal to the two right angles

ABK, BAK , which [by 17. 1.] is impossible: Therefore the planes CD, EF produced will not meet. Wherefore the planes CD, EF are parallel.

Therefore those planes to which the same right line is perpendicular, are parallel. Which was to be demonstrated.



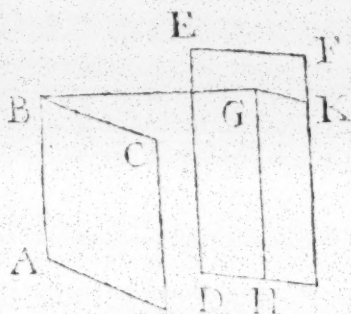
PROP. XV. THEOR.

If two right lines touching one another be parallel to two right lines touching one another, but not in the same plane; the planes that pass thro' them will also be parallel.

For let two right lines AB , BC touching one another be parallel to two right lines DE , EF touching one another, but not in the same plane: I say the planes which pass thro' AB , BC , DE , EF if produced will not meet.

For [by 11. 11.] draw BG from the point B perpendicular to the plane passing thro' DE , EF , meeting the plane in G ; and thro' G [by 31. 1.] draw GH parallel to ED , and GK to EF . Then because

BG is perpendicular to the plane passing thro' DE , EF , and [by 3. def. 11.] it will be at right angles to all the right lines that touch it, and are in that plane; and GH , GK , which are both in the plane passing thro' DE , EF , do touch it: Therefore the angles BGH , BGK are each a



right angle. And because BA is parallel to GH , the angles GBA , BGH [by 29. 1.] are equal to two right angles. But BGH is a right angle; therefore GBA will also be a right angle; and so GB is perpendicular to BA . By the same reason BG is also perpendicular to BC . Therefore because the right line BG stands at right angles to the two right lines BA , BC cutting one another: [by 4. 11.] BG also is perpendicular to the plane passing thro' AB , BC . [* and by the same reason BG is perpendicular to the plane passing thro' GH , GK . But the plane passing thro' GH , GK , is the same as that passing thro' DE , EF : Wherefore BG is perpendicular to the plane passing thro' DE , EF . But it has been proved that BG also is perpendicular to the plane passing thro' AB , BC .] and it is perpendicular to the plane passing thro' DE , EF : Therefore BG is perpendicular to both the planes passing thro' AB ,

* The words within the braces, ought to be omitted.

BC, DE, EF . But those planes to which the same right line is perpendicular, [by 14. 11.] are parallel: Therefore the planes passing thro' $AB, BC; DE, EF$ are parallel.

Therefore if two right lines touching one another be parallel to two right lines touching one another, but not in the same plane; those planes which pass thro' them will be parallel. Which was to be demonstrated.

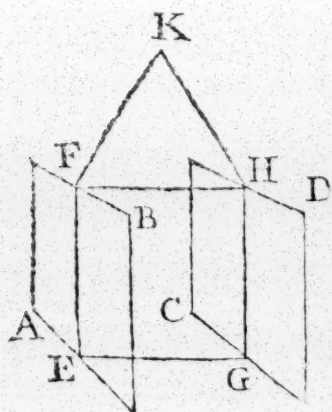
PROP. XVI. THEOR.

If any plane cuts two parallel planes, their common sections are also parallel.

For let some plane $EFGH$ cut the two parallel planes AB, CD , and let their common sections be EF, GH : I say EF is parallel to GH .

For if they be not parallel, EF, GH being produced will either meet towards F, H , or towards E, G . Let them first be produced and meet towards F, H , viz. at K : Then because EFK is in the plane AB , all the points taken in EFK will be in the same plane. But K is one of the points in EFK : Therefore K is in the plane AB : by the same reason K is also in the plane CD : Therefore the planes AB, CD being produced will meet. But they are supposed not to meet, because they are parallel: Therefore the right lines EF, GH being produced will not meet towards F, H . In like manner we demonstrate that the right lines EF, GH being produced, will not meet towards E, G . But those lines which being both ways produced, do not meet, are [by 35. def. 1.] parallel: Therefore EF, GH are parallel.

Therefore if any plane cuts two parallel planes; their common sections will be parallel. Which was to be demonstrated.



PROP. XVII. THEOR.

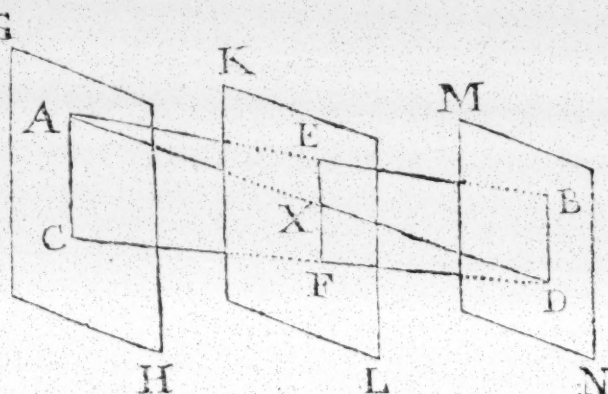
If two right lines be cut by parallel planes, they will be cut in the same ratio.

For let two right lines AB , CD be cut by the parallel planes GH , KL , MN in the points A , E , B , C , F , D : I say, as the right line AE is to EB , so is CF to FD .

For join AC , BD , AD , let AD meet the plane KL in the point X , and join EX , XF .

Then because the two parallel planes KL , MN are cut by the plane $EBDX$; [by 16. 11.] their common sections EX , BD are pa-

rallel. By the same reason because the two parallel planes GH , KL are cut by the plane $AXFC$, their common sections AC , FX are parallel: But



because EX is drawn parallel to one side BD of the triangle ABD , [by 2. 6.] as AE is to EB , so will AX be to XD . Again, because XF is drawn parallel to one side AC of the triangle ADC ; it will be as AX to XD , so is CF to FD . But it has been proved that as AX is to XD , so is AE to EB : Therefore [by 11. 5.] as AE is to EB , so is CF to FD .

Therefore if two right lines be cut by parallel planes, they will be cut in the same ratio. Which was to be demonstrated.

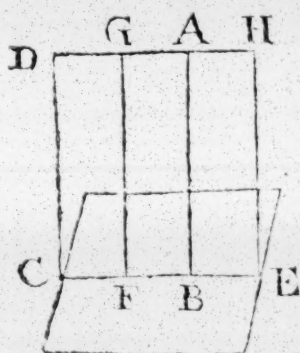
PROP. XVIII. THEOR.

If a right line be at right angles to any plane, all the planes that pass thro' it will be at right angles to that plane.

For let the right line AB be at right angles to some given plane: I say all planes that pass thro' AB will be at right angles to that given plane.

For

For let the plane DE pass thro' the right line AB , and let the right line CE be the common section of the plane DE and the given plane: Take any point F in CE , and



from the same draw FG in the plane DE at right angles to CE . Then because the right line AB is perpendicular to the given plane, it will also [by 3. def. 11.] be perpendicular to all right lines lying in the given plane and touching it: Wherefore it is also perpendicular to CE ; and so ABF is a right angle. But GFB is also a right angle: Therefore [by

28. 1.] AB is parallel to FG . But AB is at right angles to the given plane: Wherefore [by 8. 11.] FG also will be at right angles to that same plane. But [by 4. def. 11.] one plane is at right angles to another plane, when right lines drawn at right angles to their common section in one plane, are at right angles to the other plane: Therefore the right line FG drawn in one plane DE at right angles to the common section CE is proved to be at right angles to the given plane. Consequently the plane DE is at right angles to the given plane. After the same manner we demonstrate that every plane passing thro' AB is perpendicular to the given plane.

If therefore a right line be at right angles to any given plane, all the planes that pass thro' it will be also at right angles to that given plane. Which was to be demonstrated.

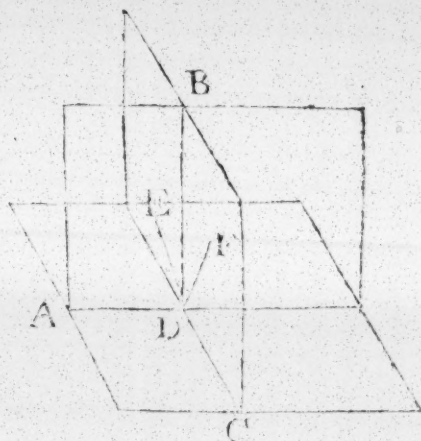
PROP. XIX. THEOR.

If two planes cutting one another be at right angles to some plane; their common section will be at right angles to the same plane.

For let two planes AB, BC , cutting one another, be at right angles to some given plane; and let their common section be BD : I say the right line BD is at right angles to that given plane.

For if it be not, [by 11. 1.] draw the right line DE from the point D in the plane AB at right angles to the right line AD , and in the plane BC draw DF at right angles

gles to CD . Then because the plane AB is at right angles to the given plane, and DE is drawn in the plane AB at right angles to their common section AD ; DE will be perpendicular to the given plane. After the same manner we demonstrate, that DF is also perpendicular to the given plane: Therefore there are two right lines drawn, on the same side, from the same point D at right angles to the given plane, which [by 13.



11.] is impossible. Therefore no right line but DB , the common section of the planes AB , CD , can be drawn from the point D at right angles to the given plane.

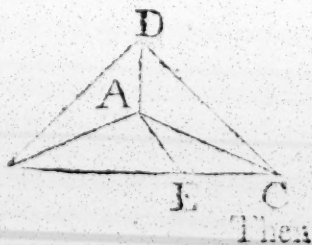
Therefore if two planes cutting one another, be at right angles to some plane; their common section will also be at right angles to that same plane. Which was to be demonstrated.

PROP. XX. THEOR.

If a solid angle be contained under three plane angles, any two of them taken together, are greater than the third,

For let a solid angle at A be contained under the three plane angles EAC , CAD , DAB : I say any two of these angles BAC , CAD , DAB , taken together, are greater than the third.

For if the angles BAC , CAD , DAB be equal to one another, it is manifest that any two of them, taken together, are greater than the third. If not, let BAC be the greatest of the three angles; and at the right line AB , and at the point A in it [by 23. 1.] make the angle BAE in the plane passing thro' BAC equal to the angle DAB ; and [by 3. 1.] make AE equal to AD ; also let the right line BE drawn thro' E cut the right lines AB , AC in the points B , C , and join DE , DC . Then



Then because DA is equal to AE , and AB is common, the two sides DA, AB are equal to the two sides AE, AB , and the angle DAB is equal to the angle BAE : Therefore [by 4. 1.] the base DB is equal to the base BE . And because the two sides DB, DC are greater than BC , but DB has been proved to be equal to BE ; the remainder DC will be greater than the remainder EC . And because DA is equal to AE , and AC is common, and the base DC is greater than the base EC ; [by 25. 1.] the angle DAC will be greater than the angle EAC . But the angle DAB has been proved to be equal to the angle BAE ; wherefore the angles DAB, DAC , taken together, are greater than the angle BAC . Also after the same manner we demonstrate, if any two other angles be taken, they are both together greater than the third angle remaining.

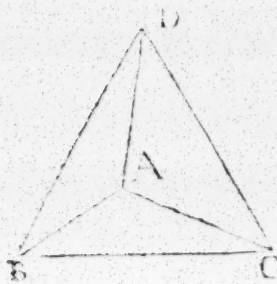
Therefore if a solid angle be contained under three plane angles, any two of them, taken together, are greater than the third. Which was to be demonstrated.

PROP. XXI. THEOR.

Every solid angle is contained under plane angles that [taken together] are less than four right angles.

Let the solid angle at A be contained under the plane angles BAC, CAD, DAB : I say the angles BAC, CAD, DAB , taken together, are less than four right angles.

For in each of the right lines AB, AC, AD , take any points B, C, D , and join BC, CD, DB . Then because the solid angle at B is contained under the three plane angles CBA, ABD, CBD ; any two of them, taken together [by 20. 11.] are greater than the third: Therefore the angles CBA, ABD are greater than the angle CBD . By the same reason the angles BCA, ACD , taken together, are greater than the angle BCD ; also the angles CDA, ADB are greater than the angle CDB : Wherefore the six angles $CBA, ABD, BCA, ACD, ADC, ADB$ are greater than the three angles CBD, BCD, CDB . But [by 32. 1.] the three angles CBD, BCD, CDB are equal to two right angles: Therefore the six angles $CBA, ABD, BCA, ACD, ADC, ADB$ are greater than two right angles. But because the three angles



angles of each of the triangles ABC , ACD , ADB are equal to two right angles, the nine angles of the three triangles CBA , ACB , BAC , ACD , DAC , CDA , ADB , DBA , BAD , are equal to six right angles. But six of the angles, *viz.* ABC , BCA , ACD , CDA , ADB , DBA , are greater than two right angles: Therefore the three remaining angles BAC , CAD , DAB , which contain the solid angle, will be less than four right angles.

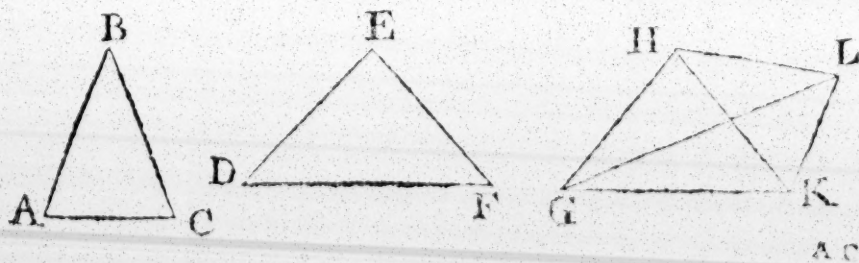
Wherefore every solid angle is contained under plane angles [which taken together] are less than four right angles. Which was to be demonstrated.

PROP. XXII. THEOR.

If there be three plane angles, any two of which taken together are greater than the third, and the lines containing these angles be equal; a triangle may be made of the three right lines joining those equal right lines.

Let there be three plane angles ABC , DEF , GHK , any two of which taken together, *viz.* the angles ABC , DEF are greater than the angle GHK , the angles DEF , GHK greater than the angle ABC , and the angles GHK , ABC greater than the angle DEF ; and let the right lines AB , BC , DE , EF , GH , HK be equal; and join AC , DF , GK ; I say, a triangle may be made of three right lines equal to AC , DF , GK ; that is, any two of these right lines AC , DF , GK , taken together, are greater than the third.

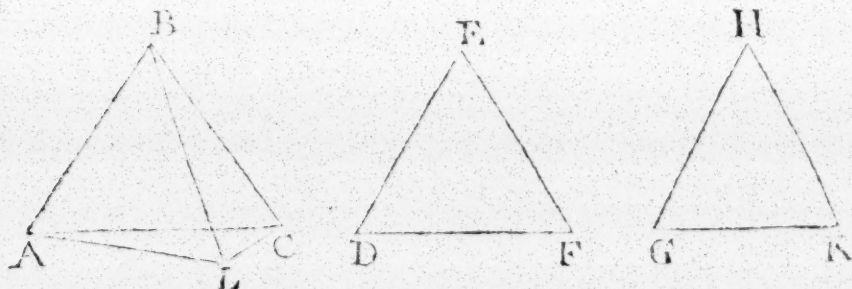
If the angles ABC , DEF , GHK are equal to one another; it is manifest a triangle may be made of three right lines equal to AC , DF , GK . But if not, let them be unequal, and at the right line HK , and at the point H in it, [by 23. I.] make the angle KHL equal to the angle ABC ; and make HL equal to either AB , BC , DE , EF , GH , or HK ; and join GL , KL . Then because the two sides AB , BC are equal to the two sides KH , HL , and the angle at B is equal to the angle KHL ; [by 4. I.] the base



AC is equal to the base KL . And because the angles ABC , $G HK$ together are greater than the angle DEF , and the angle ABC is equal to KHL ; the angle GHL will be greater than the angle DEF . Also because the two sides GH , HL are equal to the two sides DE , EF , and the angle GHL is greater than the angle E ; the base GL [by 24. 1.] will be greater than the base DF . But [by 20. 1.] GK , KL together are greater than GL : Therefore GK , KL are much greater than DF . But KL is equal to AC : Wherefore AC , GK together are greater than the remainder DF . After the same manner we demonstrate that AC , DF are greater than GK , and GK , DF greater than AC . Therefore [by 22. 1.] a triangle may be made of three right lines each equal to AC , DF , GK .

Otherwise.

Let the three given plane angles be ABC , DEF , $G HK$, any two of which, taken together, are greater than the third; let the equal right lines AB , BC , DE , EF , GH , HK contain them, and join AC , DF , GK : I say a triangle may be made of three right lines equal to AC , DF , GK , that is, any two of these right lines, taken together, are greater than the third. Therefore if again the angles at the points B , E , H are equal, the right lines AC , DF , GK [by 4. 1.] will be equal, and any two of them will be greater than the third. But if not, let the angles at the points B , E , H be unequal, and let that which is at B be



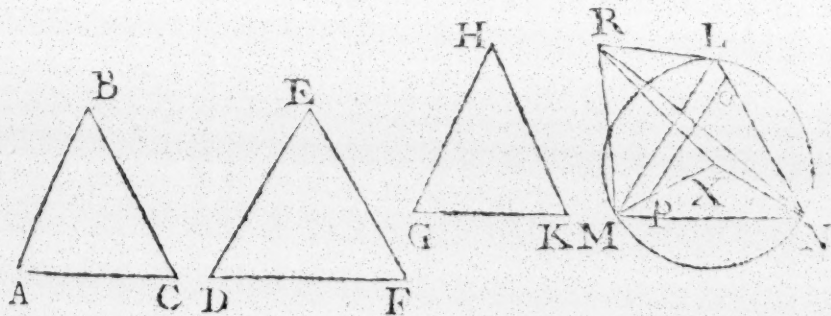
greater than that at E or H : Then [by 24. 1.] the right line AC will be greater than DF or GK ; and it is manifest that AC , together with DF or GK is greater than the remaining line: I say also that DF , GK together are greater than AC . At the right line AB , and at the point B in it [by 23. 1.] make the angle ABL equal to the angle $G HK$, make BL equal to either AB , BC , DE , EF , GH , or HK , and join AL , AC . Then because the two sides AB , BL are equal to the two sides GH , HK , each

each to each, and they contain equal angles; [by 4. 1.] the base AL will be equal to the base GK . And because the angles at the points E, H , are greater than the angle ABC , but the angle GHK is equal to the angle ABL ; the remaining angle at E will be greater than the angle LCB . Also because the two sides AB, BC are equal to the two sides DE, EF , each to each, and the angle DEF is greater than the angle LCB ; the base DF [by 24. 1.] will be greater than the base LC . But GK has been proved to be equal to AL : Therefore DF, GK are greater than AL, LC . But [by 20. 1.] AL, LC are greater than AC : Therefore DF, GK will be much greater than AC . Wherefore any two of the right lines AC, DF, GK are greater than the third; and accordingly [by 12. 1.] a triangle may be made of three right lines equal to AC, DF, GK . Which was to be demonstrated.

PROP. XXIII. PROBL.

To make a solid angle of three plane angles, any two of which, taken together, are greater than the third: But the three plane angles, taken together, must be less than four right angles.

Let the three given plane angles be ABC, DEF, GHK , any two of which, taken together, are greater than the third, and all of them together less than four right angles:



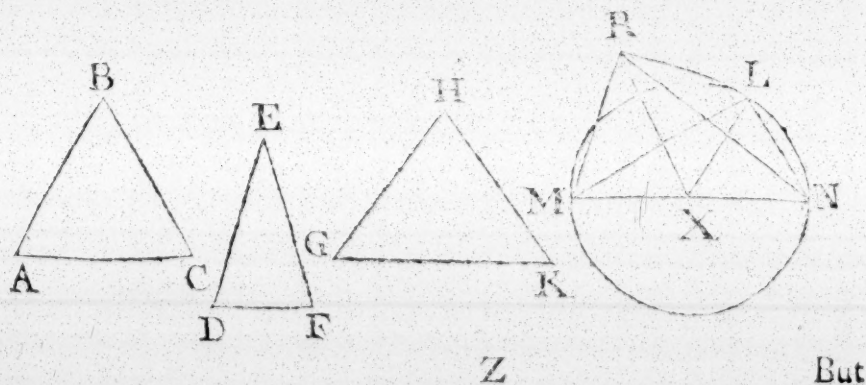
It is required to make a solid angle of three angles equal to ABC, DEF, GHK .

Cut off the equal right lines AB, BC, DE, EF, GH, HK ; and join AC, DF, GK . Then [by 22. 11.] a triangle may be made of three right lines equal to AC, DF, GK . Let this triangle as LMN be made [by 22. 1.] so that AC be equal to LM , DF to MN , and GK to LN :
Then

Then [by 5. 4] describe a circle LMN about the triangle LMN . Find the centre of this circle, which will either be within the triangle LMN , in one of its sides, or out of it.

First let the centre x of that circle be within the triangle, and join Lx , Mx , Nx : I say AB is greater than Lx . For if it be not, AB will either be equal to Lx , or less than it. First let it be equal; then because AB is equal to Lx , and AB is equal to BC ; Lx will be equal to BC . But Lx is equal to xM . Therefore the two sides AB , BC are equal to the two sides Lx , xM , each to each; and the base AC is made equal to the base LM : Wherefore [by 8. 1.] the angle ABC is equal to the angle LxM . By the same reason the angle DEF also is equal to the angle MxN , and the angle GHK to the angle NxL : Therefore the three angles ABC , DEF , GHK , are equal to the three angles LxM , MxN , NxL . But the three angles LxM , MxN , NxL are equal to four right angles: Therefore the three angles ABC , DEF , GHK will be equal to four right angles. But they are supposed to be less than four right angles, which is absurd: Therefore AB is not equal to Lx . I say also that AB is not less than Lx . For if possible let it be less; and make xO equal to AB ; xP equal to BC ; and join OP . Then because AB is equal to BC ; xO will be equal to xP : Therefore the remainder OL is equal to the remainder PM : and so [by 2. 6.] LM is parallel to OP , and the triangle LMx equiangular to the triangle OPx : Therefore [by 4. 6.] as xL is to LM , so is xO to OP ; and alternately [by 16. 5.] as Lx is to xO , so is LM to OP . But Lx is greater than xO ; therefore LM is also greater than OP . But [by constr.] LM is equal to AC : Wherefore AC will be greater than OP : Therefore because the two sides AB , BC are equal to the two sides ox , xP , and the base AC is greater than the base OP ; the angle ABC [by 24. 1.] will be greater than the angle oxP . In like manner we demonstrate, that the angle DEF also is greater than the angle MxN , and the angle GHK , greater than the angle NxL : Therefore the three angles ABC , DEF , GHK are greater than the three angles LxM , MxN , NxL . But the angles ABC , DEF , GHK are supposed to be less than four right angles: Therefore the angles LxM , MxN , NxL will be much less than
four

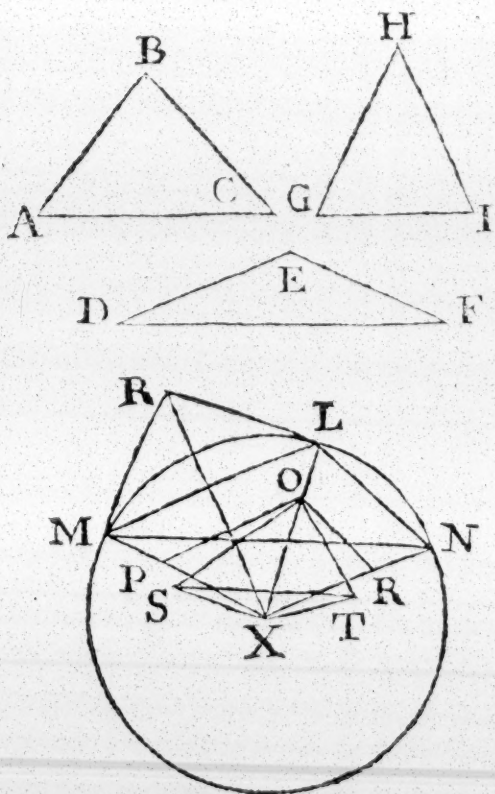
four right angles. But they are also equal to four right angles. Which is absurd. Therefore AB is not less than LX : It has been also proved to be not equal to it. Wherefore AB is greater than LX . [by the 12. 11.] raise the right line xR from the point x at right angles to the plane of the circle LMN , and make the square of xR equal to the excess whereby the square of AB exceeds the square of LX [by lemm. foll.], and join RL , RM , RN . Then because xx is perpendicular to the plane of the circle LMN , it will be [by def. 3. 11.] perpendicular to every one of the right lines Lx , Mx , Nx . And because Lx is equal to xM , and xR is common and at right angles; the base LR [by 4. 1.] will be equal to the base RM . By the same reason RN is also equal to RL or RM . Therefore the three right lines RL , RM , RN are equal to one another. And because the square of xR is made equal to the excess whereby the square of AB exceeds that of Lx ; the square of AB is equal to the squares of Lx , xR . But [by 47. 1.] the square of RL is equal to the squares of Lx , xR . For the angle AXR is a right angle. Therefore the square of AB will be equal to the square of RL , and so AB is equal to RL . But each of the lines BC , DE , EF , GH , HK is equal to AB , and RM , RN are each equal to RL . Wherefore each of the right lines AB , BC , DE , EF , GH , HK is equal to each of the right lines RL , RM , RN . And because the two sides LR , RM are equal to the two sides AB , BC , and the base LM [by contr.] is equal to the base AC : The angle LRM [by 8. 1.] will be equal to the angle ABC . By the same reason the angle MRN is equal to the angle DEF , and the angle LRN equal to the angle GHK . Therefore the solid angle at R is made of three plane angles ARM , MRN , LRN which are equal to the three given angles ABC , DEF , GHK .



But now let the centre x of the circle be in one side MN of the triangle, and join xL : I say again that AB is greater than Lx . For if it be not, AB is either equal to Lx , or less than it. First let it be equal: Then the two sides AB, BC , that is, DE, EF are equal to the two sides MX, XL , that is, to MN . But MN is put equal to DF : Therefore DE, EF are equal to DF : which [by 20. 1.] cannot be: Therefore AB is not equal to Lx ; nor likewise is it less: For then a greater absurdity would follow. Therefore AB is greater than Lx . And if after the same manner as above, Rx be drawn at right angles to the plane of the circle such that its square be equal to the excess of the square of AB above the square of Rx [by the subjoined lemma] the problem will be resolved.

Again let the centre x of the circle fall without the triangle LMN , and join Lx, Mx, Nx . I say in this case AB is greater than Lx . For if it be not, it is either equal to it, or less. First let it be equal: Then the two sides AB, BC are equal to the two sides MX, XL , each to each, and the base AC is also equal to the base ML : Therefore [by 8. 1.] the angle ABC is equal to the angle MXL . By the same reason the angle $G HK$ is also equal to the angle LxN . Wherefore the whole angle MXN is equal to the

two angles $ABC, G HK$. But the angles $ABC, G HK$ are greater than the angle DEF ; and therefore the angle MXN is greater than DEF . And because the two sides DE, EF are equal to the two sides MX, xN , and the base DF is equal to the base MN ; [by 8. 1.] the angle MXN will be equal to the angle DEF . But it has been also proved to be greater: which is absurd: therefore AB is not equal to Lx . We shall presently demonstrate that it is

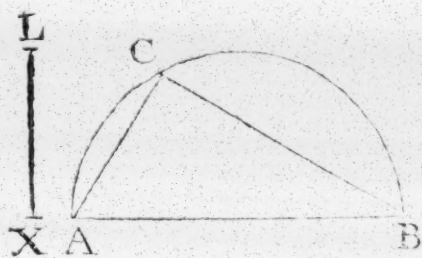


is not less : wherefore it must necessarily be greater. And if again xR be drawn at right angles to the plane of the circle such, [by the subjoined lemma] that its square be equal to the excess whereby the square of AB exceeds the square of Lx , the problem is resolved. Now I say that AB is not less than Lx . For if it be possible, let it be less, and make xO equal to AB , and xP equal to BC , and join OP . Then because AB is equal to BC ; xO will be equal to xP : Therefore the remainder OL is equal to the remainder PM : Wherefore [by 2. 6.] LM is parallel to PO ; and so the triangle LMx is equiangular to the triangle PxO : Wherefore [by 4. 6.] as Lx is to LM , so is xO to OP : and alternately as Lx is to xO , so is LM to OP . But Lx is greater than xO ; therefore LM also is greater than OP . But [by constr.] LM is equal to AC ; wherefore AC will be greater than OP ; and so because the two sides AB, BC are equal to the two sides OX, XP , each to each, and the base AC is greater than the base OP ; [by 25. 1.] the angle ABC will be greater than the angle OPX . In like manner if xR be taken equal to xO , or xP , and OR be joined, we demonstrate that the angle GHK is greater than the angle OPR . At the right line Lx , and at the point x in it, make the angle Lxs equal to the angle ABC , and the angle LxT equal to the angle GHK ; make sx, xT each equal to OX , and join Os, OT, sT . Then because the two sides AB, BC are equal to the two sides OX, xs , and the angle ABC is equal to the angle OXs , the base LC or LM [by 4. 1.] will be equal to the base Os . By the same reason LN is also equal to OT . And because the two sides ML, LN are equal to the two sides Os, OT ; and the angle MLN is greater than the angle SoT ; the base MN [by 24. 1.] is greater than the base sT . But MN is equal to DF : Therefore DF will be greater than sT . Therefore because the two sides DE, EF are equal to the two sides sx, xT , and the base DF is greater than the base sT , [by 25. 1.] the angle DEF is greater than the angle sXT . But the angle sXT is equal to the angles ABC, GHK : Therefore the angle DEF is greater than the angles ABC, GHK : but it is less too: which is impossible.

L E M M A.

We thus demonstrate how $R X$ may be so taken, that its square shall be equal to the excess of the square of $A B$, above $L X$.

Let the right lines be $A B$, $L X$, and let $A B$ be the greater. Upon $A B$ describe the semicircle $A B C$: And in $A B C$ apply the right line $A C$ equal to $L X$, and join $B C$.



Then because the angle $A C B$ is in the semicircle $A B C$, the angle $A C B$ [by 31. 3.] will be a right angle. Therefore the square of $A B$ [by 47. 1.] is equal to the square of $A C$, and to the square of $C B$:

Wherefore the square of $A B$ exceeds the square of $A C$ by the square of $C B$. But $A C$ is equal to $L X$: Wherefore the square of $A B$ exceeds the square of $L X$ by the square of $C B$. Wherefore if $x R$ be taken equal to $C B$, the square of $A B$ will exceed the square of $L X$ by the square $x R$. Which was the thing to be done.

◦ As the measure or relative quantity of a plane right-lined angle, is the sector of a circle contained under the sides of the angle, or the arch of a circle described about the angular point contained between the sides of the angle. So in like manner, is the measure of a solid angle, the sector of a sphere included by the plane angles, of which that solid angle consists; or, the surface of a sphere formed by the intersections of the plane angles of which a solid angle consists, all meeting at the centre of the sphere, is the measure of a solid angle.

I have often thought that the theory of solid angles may be much further extended, and perhaps made useful.

P R O P. XXIV. T H E O R.

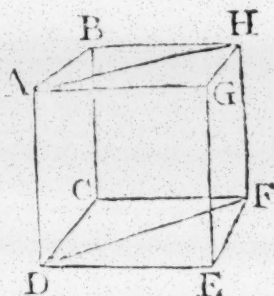
If a solid be contained under parallel planes; the opposite planes of it are equal, and parallelograms^P.

For let the solid $C D G H$ be contained under the parallel planes $A C$, $G F$, $A H$, $D F$, $F B$, $A E$: I say its opposite planes are equal and parallelograms.

For

For because the two parallel planes BG, CE are cut by the plane AC , their common sections [by 16. 11.] are parallel: Therefore AB is parallel to DC . Again, because the two parallel planes BF, AE are cut by the plane AC , their common sections are parallel: Therefore AD is parallel to BC . And because AB has been proved to be parallel to DC : Therefore AC is a parallelogram. In like manner we demonstrate that DF, FG, GB, BF, AE are each of them parallelograms.

Join AH, DF : Then because AB is parallel to DC , and BH to CF , the two right lines AB, BH touching one another will be parallel to the two right lines DC, CF touching one another, but not in the same plane: Wherefore [by 10. 11.] they will contain equal angles: Therefore the angle ABH is equal to the angle DCF . And because [by 34. 1.] the two sides AB, BH are equal to the two sides DC, CF , and the angle ABH is equal to the angle DCF ; the base AH [by 4. 1.] will be equal to the base DF , and the triangle ABH equal to the triangle DCF . And [by 34. 1.] the parallelogram BG is double to the triangle ABH , and the parallelogram CE to the triangle DCF : Therefore the parallelogram BG is equal to the parallelogram CE . In like manner we demonstrate that the parallelogram AC is equal to the parallelogram CF , and the parallelogram AE equal to the parallelogram BF .



If therefore a solid be contained under parallel planes, its opposite planes will be equal and parallelograms. Which was to be demonstrated.

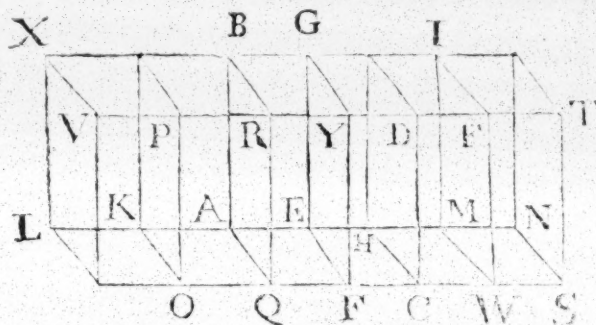
^P This proposition does not always hold true, unless the solid has six faces: for a solid may be contained under parallel planes, and each of the opposite planes will be indeed equal. But some of them may not be parallelograms; as, if the eight angles of a cube were cut off by equal parallel planes, the remaining solid would consist of eight equal and opposite triangles and eight equal and opposite parallelograms or squares. :

PROP. XXV. THEOR.

If a solid parallelepipedon be cut by a plane, parallel to its opposite planes: it will be as the base is to the base, so is the solid to the solid.

For let the solid parallelepipedon $ABCD$ be cut by the plane YE parallel to the opposite planes RA , DH : I say as the base $AEFQ$ is to the base $EHCF$, so is the solid $ABFY$ to the solid $EGCD$.

For produce AH both ways, and take any number of parts HM , MN each equal to LH , and any others AK , KL each equal to AE , and compleat the parallelograms LO , KQ , HW , MS , and the solids LP , KR , DM , MT . Then



because the right lines LK , KA , AE are equal to one another, the parallelograms LO , KQ , AF , [by 38. 1.] will be equal to one another; as also the parallelo-

grams KX , KB , AG ; moreover the parallelograms LV , KP , AR are [by 24. 11.] equal to one another; for they are opposite parallelograms. By the same reason the parallelograms EC , HW , MS are equal to one another; as also the parallelograms HG , HI , IN , are equal to one another: and so are the parallelograms DH , MF , NT : Therefore three planes of the solids LP , KR , AY are equal to three planes. But these three planes are equal to the three opposite ones: Therefore [by 10. def. 11.] the three solids LP , KR , AY will be equal to one another: By the same reason the three solids ED , DM , MT are equal to one another: Therefore the base LF is the same multiple of the base AF , as the solid LY is of the solid AY . By the same reason, the base NF is the same multiple of the base EF , as the solid NY is of the solid HY . Again if the base LF be equal to the base NF , the solid LY will be equal to the solid NY , and if LF exceeds the base NF , the solid LY will exceed the solid NY ; and if less, less. There are therefore

therefore four magnitudes, *viz.* the two bases AF, FH , and the two solids AY, YH . and there are taken equimultiples of the base AF , and the solid AY , *viz.* the base LF , and the solid LY , and the base NF , and the solid NY : And it has been demonstrated, if the base LF exceeds the base NF , the solid LY does exceed the solid NY ; and if it be equal, equal; if less, less. Therefore [by 5. def. 5.] as the base AF is to the base FH , so is the solid AY to the solid YH . Which was to be demonstrated.

¶ The demonstration of this theorem is another proof of the existence and truth of the fifth definition of the fifth book.

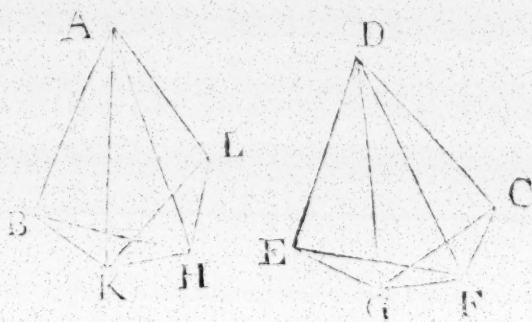
PROP. XXVI. PROBL.

To make a solid angle at a given right line, and at a given point in it, equal to a given solid angle^r.

Let the given right line be AB , and A the given point in it: and let the given solid angle be at D , contained under the three plane angles EDC, EDF, FDC : it is required to make a solid angle at the given right line AB , and at the given point A in it, equal to the solid angle at D .

Take any point F in the line DF , from which [by II. 11.] draw FG perpendicular to the plane passing thro' ED, DC , meeting the plane in the point G , and join DG : Again make [by 23. 1.] the angle BAL , at the given right line AB , and at the given point A in it, equal to the angle EDC , and the angle BAK equal to the angle EDG : and [by 3. 1.] make AK equal to DG , and [by 12. 11.] draw KH from K at right angles to the plane passing thro' BAL , and

make KH equal to GF , and join HA : I say the solid angle at A , which is contained under the angles BAL, BAH, HAL is equal to the solid angle at D comprehended under the angles EDC, EDF, FDC .



For take the equal right lines AB, DE ; and join HB, KB, FE, GE . Then because FG is perpendicular to the plane drawn thro' ED, DC ; it will be perpendicular to all right lines that touch it, and are in that plane [by 3. def. II.]: Therefore the angles FGD, FGE are each of them a right angle. By the same reason the angles HKA, HKB are each of them a right angle. And because the two sides KA, AB are equal to the two sides GD, DE , each to each, and they contain equal angles; the base BK will be [by 4. I.] equal to the base EG . But KH is equal to GF , and they contain right angles: Therefore HB is equal to FE . Again, because the two sides AK, KH are equal to the two sides DG, GF , and they contain equal angles, the base AH will be equal to DF . But AB is equal to DE : Therefore the two sides HA, AB are equal to the two sides FD, DE , and the base HB is equal to the base FE : Therefore the angle BAH will be equal to the angle EDF : By the same reason the angle HAL is equal to the angle FDC , viz. if we take AL, DC equal, and join KL, HL, GC, FC , because the whole angle BAL is equal to the whole angle EDC , and BAK is put equal to EDG ; the remaining angle KAL will be equal to the remaining angle GDC . And because the two sides KA, AL are equal to the two sides GD, DC , and they contain equal angles; the base KL [by 4. I.] will be equal to the base GC . But KH is equal to GF : Therefore the two sides LK, KH are equal to the two sides CG, GF , and they contain right angles; wherefore the base HL is equal to the base FC . Again, because the two sides HA, AL are equal to the two sides FD, DC , and the base HL is equal to the base FC ; the angle HAL [by 8. I.] will be equal to the angle FDC , and the angle BAL equal to the angle EDC .

Therefore a solid angle is made at a given right line, and at a given point in it, equal to a given solid angle. Which was to be done.

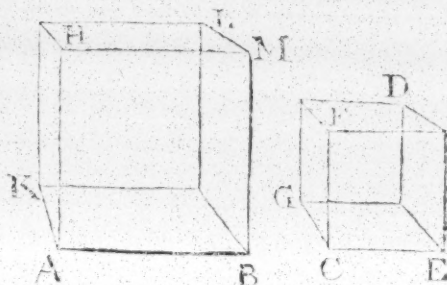
The solution of the following problem would be new, viz. a solid angle consisting of three equal plane angles being given: To find a solid angle consisting of four equal plane angles, that shall be equal to it, when possible.

PROP. XXVII. PROBL.

Upon a given right line to describe a solid parallelepipedon similar and alike situate to a given solid parallelepipedon^s.

Let the given right line be AB , and the given solid parallelepipedon be DC : It is required to describe a parallelepipedon upon the given right line AB , similar and alike situate to the given solid parallelepipedon DC .

For at the given right line AB , and at the given point A in it, make the solid angle which is contained under the plane angles BAH , HAK , KAB , equal to the solid angle C , so that the angle BAH , be equal to the angle ECF , the angle BAK equal to ECG , and the angle KAH equal to the angle GCF . Then [by 12. 6.] make as EC is to CG , so is BA to AK , and as GC is to CF , so is KA to AH . Then by equality of ratio [by 22. 5.] as EC is to CF , so will BA be to AH . Lastly, complete the parallelogram BH and the solid AL .



Then because it is as EC is to CG , so is BA to AK , the sides about the equal angles ECG , BAK are proportional: and so [by 4. 6.] the parallelogram GE is similar to the parallelogram KB . Also by the same reason, the parallelogram KH is similar to the parallelogram GF , and the parallelogram FE to the parallelogram HB : Therefore three parallelograms of the solid CD , are similar to three parallelograms of the solid AL . But [by 24. 11.] the three opposite parallelograms are equal and similar to them. Therefore the whole solid CD will be similar to the whole solid AL .

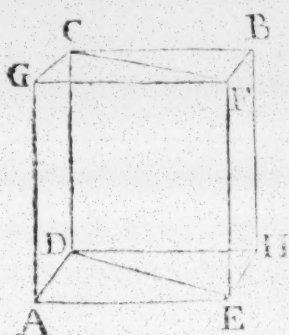
Therefore upon a given right line AB a solid parallelepipedon AL is described similar to a given solid parallelepipedon CD , and alike situate. Which was to be done.

^s Two solids are said to be alike situate, when their homologous or answerable planes have like situations.

PROP. XXVIII. THEOR.

If a solid parallelepipedon be cut by a plane passing thro' the diagonals of the opposite parallelograms; the solid will be cut by that plane into two equal parts.

For let the solid parallelepipedon AB be cut by the plane $CDEF$ passing thro' the diagonals CF , DE of two opposite planes: I say the solid AB is cut in half by the plane $CDEF$.



For because [by 34. 1.] the triangle CGF is equal to the triangle CBF , and the triangle ADE to the triangle DEH , and [by 24. 11.] the parallelogram CA is equal to the parallelogram BE , for it is opposite: and the parallelogram GE is equal to the parallelogram CH : [by 10. def. 11.] the prism contained under the two triangles CGF , ADE , and the three parallelograms GE , AC , CE is equal to the prism contained under the two triangles CBF , DEH , and the three parallelograms CH , BE , CE : for they are contained under equal numbers of equal planes.

Therefore the whole solid AB is cut in half by the plane $CDEF$. Which was to be demonstrated.

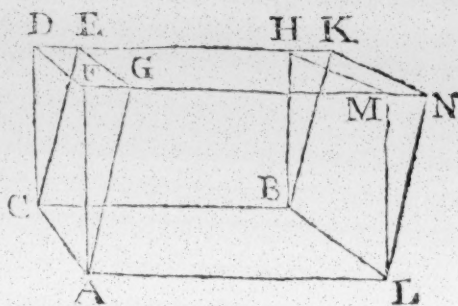
PROP. XXIX. THEOR.

Solid parallelepipedons, standing upon the same base, and having the same altitude, and whose sides standing upon the common base, terminate in the same right lines, are equal to one another.

For let the solid parallelepipedons CM , CN , of the same altitude, stand upon the same base AB , and let their sides AF , AG , LM , LN ; CD , CE , BH , BK terminate in the right lines FN , DK : I say the solid CM is equal to the solid CN .

For because CH , CK are both parallelograms: [by 34. 1.] CB will be equal to DH , or EK ; therefore DH is equal to EK . Take away EH , which is common; then the

the remainder DE is equal to the remainder HK ; and so [by 8. 1.] the triangle DEC is equal to the triangle HKB . But [by 36. 1.] the parallelogram DG is equal to the parallelogram HN . By the same reason the triangle



AFG is equal to the triangle LMN . But [by 24. 11.] the parallelogram CF is equal to the parallelogram BM , and the parallelogram CG to the parallelogram BN ; for they are opposite. Therefore [by 10. def. 11] the prism contained under the two triangles AFG , DEC , and the three parallelograms AD , GD , GC is equal to the prism contained under the two triangles LMN , HBK , and the three parallelograms BM , NH , BN . Add the common solid, whose base is the parallelogram AB , and its opposite is $GEHM$: Therefore the whole solid parallelepipedon CM is equal to the whole solid parallelepipedon CN .

Therefore solid parallelepipedons upon the same base, and having the same altitude, and whose sides standing upon the common base, terminate in the same right lines, are equal to one another. Which was to be demonstrated.

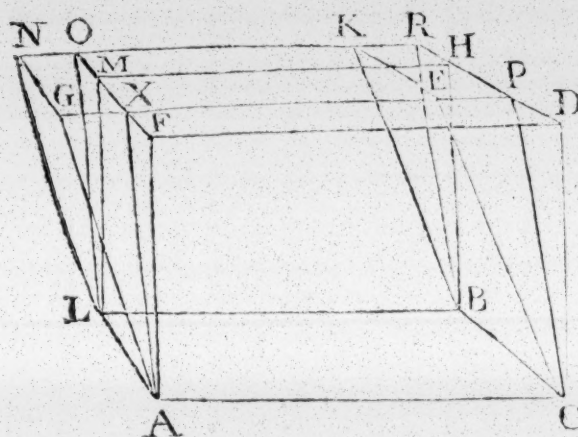
PROP. XXX. THEOR.

Solid parallelepipedons, having the same base and altitude, and whose sides standing upon the common base, do not terminate in the same right lines, are equal to one another.

For let the solid parallelepipedons CM , CN be upon the same base AB , and have the same altitude, and let their sides AF , AG , LM , LN , CD , CE , BH , BK , standing upon the base not terminate in the same right lines: I say the solid CM is equal to the solid CN .

For produce NK , DH ; as also GE , FM , and let them meet in the points O , R , P , X ; and join AX , LO , CP , BR .

The solid CM , whose base is the parallelogram $ACBL$, and opposite parallelogram $FEHM$ is [by 29. 11.] equal to the solid CO , whose base is the parallelogram $ACBL$, and



and opposite parallelogram $xpro$; for they stand upon the same base $acbl$, and their sides standing upon the base af , ax , lm , lo , cd , cp , bh , br do terminate in the same right lines fo , dr . But the

solid co , whose base is the parallelogram $acbl$, and the parallelogram $xpro$ opposite to it, is [by 29. 11.] equal to the solid cn , whose base is the parallelogram $acbl$, and parallelogram opposite to it is $gekn$; for they both have the same base $acbl$, and the sides standing upon the base ag , ax , ce , cp , ln , lo , bk , br , terminate in the same right lines gp , nr : Wherefore the solid cm is equal to the solid cn .

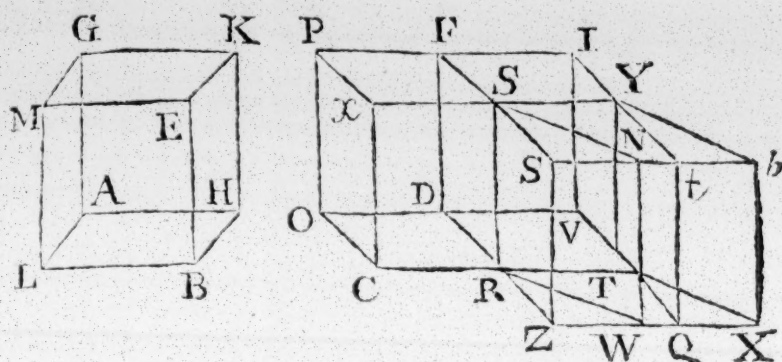
Therefore solid parallelepipeds having the same base and altitude, whose sides standing upon the common base do not terminate in the same right lines, are equal to one another. Which was to be demonstrated.

PROP. XXXI. THEOR.

Solid parallelepipeds, standing upon equal bases, and having the same altitude, are equal to one another.

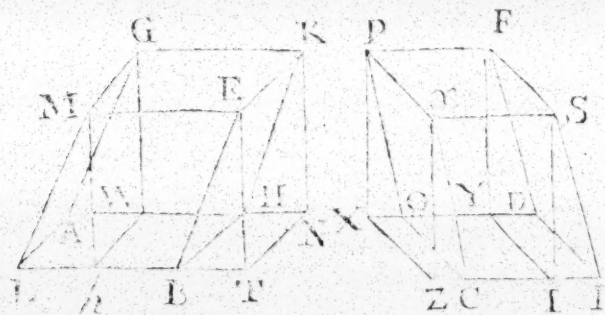
Let the solid parallelepipeds ae , cf standing upon the equal bases ab , cd have the same altitude: I say the solid ae is equal to the solid cf .

First let the sides standing upon the bases hk , be , ag , lm , op , df , cx , rs be at right angles to the bases ab , cd , and let the angle alB be unequal to the angle crD ; continue out cr directly to t , and at the right line rt , and at the point r in it, make [by 23. 1.] the angle trw , equal to the angle alB , and make rt equal to al , and rw equal to lb , and through the point w draw xw parallel to rt , compleat the base rx , and the solid yw . Then because the two sides tr , rw are equal



equal to the two sides AL , LB , and they contain equal angles, the parallelogram RX will be equal and similar to the parallelogram HL . And again because AL is equal to RT , and LM to RS , and they contain equal angles: the parallelogram RY will be equal and similar to the parallelogram AM . By the same reason the parallelogram LE is equal and similar to the parallelogram sw . Therefore three parallelograms of the solid AE are equal and similar to three parallelograms of the solid wY . But [by 24. II.] these parallelograms are equal and similar to the three opposite ones: Therefore the whole solid parallelepipedon AE is [by 10. def. II.] equal to the whole solid parallelepipedon Yw . Produce DR , xw meeting in z , and draw TQ thro' T , parallel to DZ , continue out TQ , OD meeting one another at v , and compleat the solids zy , RI : Then the solid yz , whose base is the parallelogram RY , and opposite parallelogram is zt , is [by 29. II.] equal to the solid zy , whose base is the parallelogram RY , and opposite parallelogram is wh : for they stand upon the same base RY , have the same altitude, and their sides drawn from the base RZ , RW , TQ , TX , SS , SN , Yt , Yb terminate in the same right lines zx , sh . But the solid wY , is equal to the solid AE : Therefore the solid yz is equal to the solid AE : Again, because the parallelogram $RwxT$ is [by 35. I.] equal to the parallelogram zT : for they stand upon the same base RT , and are between the same parallels RT , zx . But the parallelogram $RwxT$ is equal to the parallelogram CD ; because it is also equal to AB . Therefore the parallelogram zT is equal to the parallelogram CD . But DT is another parallelogram: Wherefore [by 7. 5.] as the base CD is to the base DT , so is zT to DT . And because the solid parallelepipedon CI is cut by

the plane RF parallel to the opposite planes; as the base CD is to the base DT , so [by 25. 11.] will the solid CF be to the solid RI . For the same reason, because the solid parallelepipedon ZI is cut by the plane RY , parallel to the opposite planes, as the base ZT is to the base DT , so will the solid ZY be to the solid RI . But as the base CD is to the base DT , so is the base ZT to the base TD : Therefore as the solid CF is to the solid RI , so [by 11. 5.] is the solid ZY to the solid RI . And since each of the solids CF , ZY has the same ratio to the solid RI , the solid CF , [by 9. 5.] is equal to the solid ZY : But the solid ZY has been proved to be equal to the solid AE . Therefore the solid AE will be equal to CF . Now let the sides standing upon the base AG , HK , BE , LM , CA , OP , DF , RS not be at right angles to the bases AB , CD : I say again the solid AE is equal to the solid CF . For [by 11. 11.] from the points K , E , G , M , P , F , x , s , draw the perpendiculars KN , ET , GW , Mb , PX , FY , xZ , SI , to the plane of the bases, meeting the planes in the points N , T ,



w , b , x , y , z , and I ; and join NT , wb , NW , Tb , xy , xz , ZI , YI . Then [by 31. 11.] the solid KNb will be equal to the solid PI ; for they

stand upon the equal bases KN , PI , and have the same altitude, and their sides standing upon the bases are at right angles to the bases. But [by 30. 11.] the solid KNb is equal to the solid AE ; and the solid PI equal to the solid CF : for they stand upon the same base, have the same altitude, and their sides standing upon the base do not terminate in the same right lines: therefore the solid AE will be equal to the solid CF .

Wherefore solid parallelepipeds standing upon the same base, and having the same altitude, are equal to one another. Which was to be demonstrated.

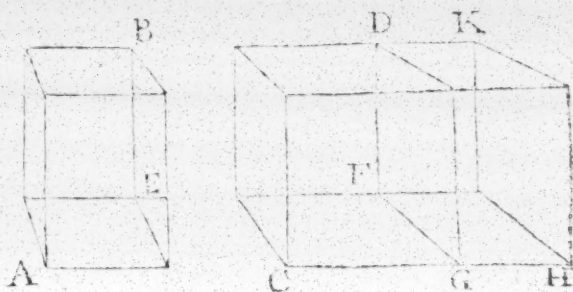
PROP. XXXII. THEOR.

Solid parallelepipeds that have the same altitude, are to one another as their bases¹.

Let the solid parallelepipeds AB, CD have the same altitude: I say they are to one another as their bases; that is, as the base AE is to the base CF , so is the solid AB to the solid CD .

For [by 45. 1.] to FG apply the parallelogram FH equal to the parallelogram AE , and upon the base FH complete the parallelepipedon GK having the same altitude as the solid CD .

Then [by 31. 11.] the solid AB is equal to the solid GK ; for they stand upon the equal bases AE, FH , and have the same altitude: Wherefore



because the solid parallelepipedon CK is cut by the plane DG , parallel to the opposite planes; it will be [by 25. 11.] as the base HF is to the base CF , so is the solid HD to the solid DC . But the base FH is equal to the base AE , and the solid GK to the solid AB : Therefore as the base AE is to the base CF , so is the solid AB to the solid CD .

Wherefore solid parallelepipeds, having the same altitude, are to one another as their bases. Which was to be demonstrated.

¹ Here is another instance of the truth and existence of the fifth definition of the fifth book.

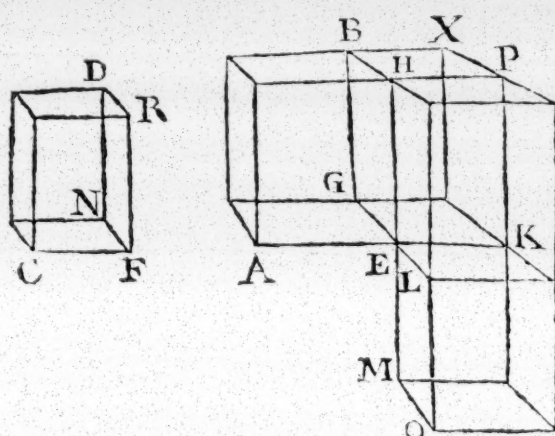
PROP. XXXIII. THEOR.

Similar solid parallelepipeds are to one another in the triplicate ratio of their homologous sides.

Let AB, CD be similar solid parallelepipeds, and the side AE homologous to the side CF : I say the solid AB has to the solid CD a ratio triplicate of the ratio of the side AE to the side CF .

For

For let EK , EL , EM be the direct continuations of AE , GE , HE , make EK equal to CF , EL equal to FN , and EM equal to FR , and compleat the solid KO , and the parallelogram KL . Then because the two sides EK , EL are equal to the two sides CF , FN , and the angle KEL is equal to the angle CFN , because the angle AEG is equal to CFN , since the solids AB , CD are similar, the parallelogram KL will be equal and similar to the parallelogram CN . By the same reason the parallelogram KM will be equal and similar to the parallelogram CR , and the parallelogram OE to the parallelogram DF : Therefore three parallelograms of the solid KO , are equal and similar to



three parallelograms of the solid CD . But [by 24. 11.] the three opposite parallelograms in each solid are equal and similar to these: Therefore the whole solid KO [by 10. def. 11.] is equal and similar to the whole

solid CD . Compleat the parallelogram GK , and upon the parallelogram bases GK , KL compleat the solids EX , LP , having the same altitude as the solid AB . Then because from the similarity of the solids AB , CD , it is as AE to CF , so is EG to FN , and EH to FR , and FC is equal to EK , and FN to EL , and FR to EM : It will be as AE is to EK , so is GE to EL , and HE to EM . But [by 1. 6.] as AE is to EK , so is the parallelogram AG to the parallelogram GK : But as GE is to EL , so is GK to KL , and as HE is to EM , so is PE to KM : Therefore as the parallelogram AG is to the parallelogram GK , so is GK to KL , and PE to KM . But [by 32. 11.] as AG is to GK , so is the solid AB to the solid EX . And as GK is to KL , so is the solid XE to the solid PL , and as PE is to KM , so is the solid PL to the solid KO : And therefore as the solid AB is to the solid EX , so is EX to PL , and PL to KO . But if there be four magnitudes continual proportionals,

nals, the first to the fourth has a ratio [by 11. def. 5.] the triplicate of the ratio that the first has to the second; and therefore the solid AB to the solid KO will be in the triplicate ratio of AB to EX . But [by 32. 11.] as AB is to EX , so is the parallelogram AG , to the parallelogram GK , and [by 1. 6.] the right line AE to EK : Wherefore the solid AB will be to the solid KO in the triplicate ratio of AE to EK . But the solid KO is equal to the solid CD , and the right line EK to the right line CF : Wherefore the solid AB has a ratio to the solid CD , triplicate of the ratio of the homologous side AE to the homologous side CF . Which was to be demonstrated.

Corollary. From hence it is manifest, if four right lines be * proportional, as the first is to the fourth, so is the solid parallelepipedon described upon the first, to a solid parallelepipedon similar and alike situate described upon the fourth; because the first line to the fourth, is in the triplicate ratio of the first to the second.

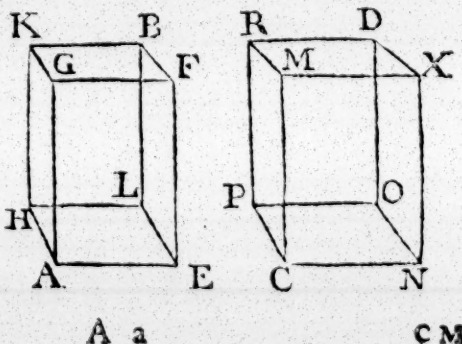
* Continual it should be.

PROP. XXXIV. THEOR.

The bases of equal solid parallelepipedons are reciprocally proportional to their altitudes. And those solid parallelepipedons whose bases are reciprocally proportional to their altitudes, are equal to one another.

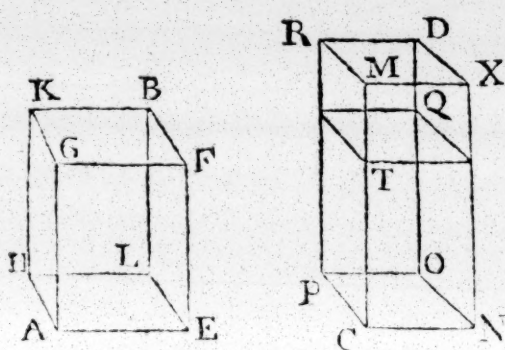
Let AB , CD be equal solid parallelepipedons: I say their bases are reciprocally proportional to their altitudes; that is, as the base EH is to the base NP , so is the altitude of the solid CD to the altitude of the solid AB .

For first let the sides standing upon the bases AG , EF , LB , HK , CM , NX , OD , PR be at right angles to the bases: I say as the base EH is to the base NP , so is CM to AG . For if the base EH be equal to the base NP , the solid AB is also equal to the solid CD ; then will



CM be equal to AG ; for if, when the bases EH , NP are equal, the altitudes AG , CM are not equal: the solid AB [by 31. 11.] will not be equal to the solid CD . But it is supposed to be equal to it: Therefore the altitude CM is not unequal to the altitude AG : Wherefore they are equal: and so the base EH will be to the base NP , as CM is to AG . Therefore it is manifest that the bases and altitudes of the solid parallelepipeds AB , CD are reciprocally proportional.

But now let the base EH be unequal to the base NP , viz. greater than it, and the solid AB is equal to the solid CD : Therefore CM is greater than AG . For if it be not,



the solids AB , CD [by 31. 11.] would again be unequal: But they are supposed to be equal: now make CT equal to AG , and upon the base NP , and altitude CT , compleat the solid parallelepipedon QC . Then be-

cause the solid AB is equal to the solid CD , and QC is another solid, and [by 7. 5.] equal magnitudes have the same ratio to the same magnitude; the solid AB will be to the solid CQ , as the solid CD is to the solid CQ . But [by 32. 11.] as the solid AB is to the solid CQ , so is the base EH to the base NP , for the altitudes of the solids AB , CQ are equal. But as the solid CD is to CQ , so [by 25. 11.] is the base MP to the base PT , and [by 1. 6.] so is MC to CT : Therefore as the base EH is to the base NP , so is the base MC to CT . But CT is equal to AG : Therefore as the base EH is to the base NP , so is MC to AG . Wherefore the bases and altitudes of the solid parallelepipeds AB , CD are reciprocally proportional.

AGAIN. Let the bases and altitudes of the solid parallelepipeds AB , CD be reciprocally proportional; and let the base EH be to the base NP , as the altitude of the solid CD is to the altitude of the solid AB : I say the solid AB is equal to the solid CD .

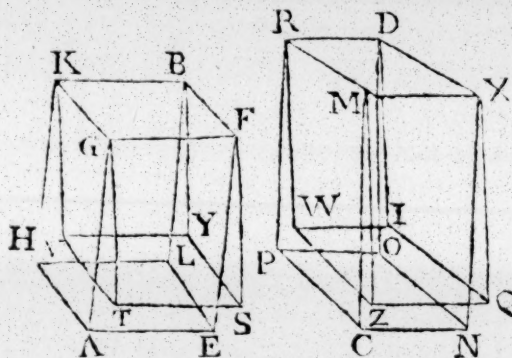
For again let the sides standing upon the bases be at right angles to the bases; and if the base EH be equal to the

the base NP ; and it is as the base EH is to the base NP , so is the altitude of the solid CD to the altitude of the solid AB ; the altitude of the solid CD will be equal to the altitude of the solid AB . But solid parallelepipeds which stand upon the same base, and have the same altitude, are [by 31. 11.] equal to one another. Therefore the solid AB is equal to the solid CD .

But now let the base EH not be equal to the base NP , viz. let it be greater; then the altitude of the solid CD is greater than the altitude of the solid AB ; that is, CM greater than AG . Again, make CT equal to AG , and compleat the solid CQ in like manner as before. Then because as the base EH is to the base NP , so is CM to AG ; and AG is equal to CT ; it will be as the base EH is to the base NP , so is MC to CT : But [by 32. 11.] as the base EH is to the base NP , so is the solid AB to the solid CQ ; for the solids AB , CQ have equal altitudes. And as MC is to CT , so [by 1. 6.] is the base MP to the base PT , and [by 25. 11.] so is the solid CD to the solid CQ . Therefore as the solid AB is to the solid CQ , so is the solid CD to the solid CQ . Therefore each of the solids AB , CD have the same ratio to the solid CQ : Wherefore [by 9. 5.] the solid AB is equal to the solid CD . Which was to be demonstrated.

But now let the sides standing upon the bases FE , BL , GA , KH , KN , DO , MC , RP not be at right angles to the bases, and [by 11. 11.] from the points F , G , B , K , X , M , D , R , draw perpendiculars to the planes of the bases EH , NP meeting the planes in the points S , T , Y , V , Q , Z , I , W , and compleat the solids FV , XW : I say and thus also when the solids AB , CD are equal, the bases are reciprocally proportional to their altitudes; and as the base EH is to the base NP , so is the altitude of the solid CD to the altitude of the solid AB . For because the solid AB is equal to the solid CD , and [by 30. 11.] the solid BT is equal to the solid AB , for they both stand upon the same base FK , have the same altitude, and their sides standing upon the bases do not terminate in the same right lines; and the solid DC is equal to the solid DZ ; for they both stand upon the same base XR , have the same altitude, and their sides standing upon the base do not terminate in the same right lines; the solid BT will be equal to the solid DZ . But

[by the part foregoing of this] the bases of equal solid parallelepipeds at right angles to their bases, are reciprocally proportional to their altitudes: Therefore as the base



FK is to the base XR , so is the altitude of the solid DZ to the altitude of the solid BT . But [by 24. 11.] the base FK is equal to the base EH , and the base XR is equal to the base NP . Wherefore as the base EH is to the base NP ,

so is the altitude of the solid DZ , to the altitude of the solid BT . But the solids DZ , BT have the same altitude as the solids DC , BA . Therefore as the base EH is to the base NP , so is the altitude of the solid DC to the altitude of the solid BA . Wherefore the bases and altitudes of the solid parallelepipeds AB , CD are reciprocally proportional.

AGAIN. Let the bases and altitudes of the solid parallelepipeds AB , CD be reciprocally proportional; and let it be as the base EH to the base NP , so is the altitude of the solid CD to the altitude of the solid AB : I say the solid AB is equal to the solid CD .

For the same construction remaining. Because as the base EH is to the base NP , so is the altitude of the solid CD to the altitude of the solid AB ; and the base EH is equal to the base FK , and NP to XR ; the base FK will be to the base XR , as the altitude of the solid CD , is to the altitude of the solid AB : for the altitudes of the solids AB , CD are the same as those of the solids BT , DZ . Therefore as the base FK is to the base XR , so is the altitude of the solid DZ , to the altitude of the solid BT : Wherefore the bases of the solid parallelepipeds BT , DZ are reciprocally proportional to their altitudes. But those solid parallelepipeds, whose sides are at right angles to their bases, and the bases are reciprocally proportional to their altitudes, are [by the former part of this] equal to one another. Therefore the solid BT is equal to the solid DZ . But [by 30. 11.] the solid BA is equal to the solid BT for they stand upon the same base FK , have the same altitudes,

the

tude, and their sides standing upon the base do not terminate in the same right lines: Also the solid DZ is equal to the solid DC ; for they stand upon the same base XR , have the same altitude, and their sides standing upon the base do not terminate in the same right lines. Therefore the solid AB is equal to the solid CD . Which was to be demonstrated.

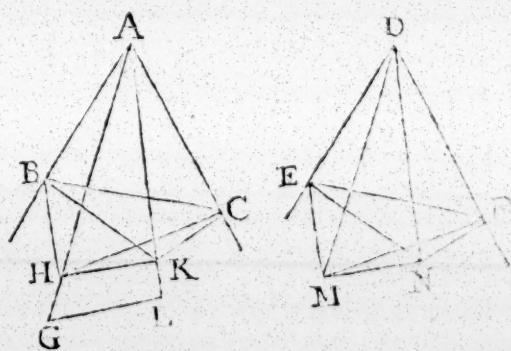
^v Such a theorem as this holds good in triangles, viz. equal triangles have their bases and altitudes reciprocally proportional; and those triangles whose bases and altitudes are reciprocally proportional, are equal.

PROP. XXXV. THEOR.

If there be two equal plane angles, and right lines be elevated from their vertexes, containing equal angles with the sides of the given plane angles, each to each; and if any points be taken in those elevated right lines, and from them be drawn perpendiculars to the planes wherein are the first mentioned angles; and from the points wherein these perpendiculars meet the planes, right lines be drawn to the first mentioned angles; these right lines will make equal angles with the elevated lines.

Let there be two equal right lined angles BAC, EDF , and let the right lines AG, DM be elevated from the points A, D , making equal angles with the sides of the first mentioned angles, each to each, the angle MDE equal to the angle GAB , and the angle MDF equal to the angle GAC ; and take any points G, M in AG, DM , from which draw the perpendiculars GL, MN to the planes passing thro' BAC, EDF , meeting those planes in the points L, N ; and join AL, ND : I say the angle GAL is equal to the angle MDN .

Make AH equal to



DM ; and thro' H draw HK parallel to GL . But GL is perpendicular to the plane passing thro' BAC : Wherefore [by 8. 11.] HK will be perpendicular to the plane passing thro' BAC . From the points K, N draw the perpendiculars KB, KC, NF, NE to the right lines AB, AC, DF, DE , and join HC, CB, MF, FE . Then because [by 47. 1.] the square of HA is equal to the squares of HK, KA , and the squares of KC, CA are equal to the square of KA ; the square of HA will be equal to the squares of HK, KC, CA : But the square of HC is equal to the squares of HK, KC : Therefore the square of HA will be equal to the squares of HC, CA ; and so the angle HCA is a right angle. By the same reason the angle DFM is a right angle: Therefore the angle ACH is equal to DFM : But the angle HAC is equal to the angle MDF : Therefore MDF, HAC are two triangles having two angles of the one equal to two angles of the other, each to each, and one side of the one equal to one side of the other, *viz.* the sides opposite to the equal angles, *viz.* AH [by constr.] equal to DM : Therefore [by 26. 1.] the remaining sides of the one will be equal to the remaining sides of the other, each to each: Wherefore AC is equal to DF . After the same manner we demonstrate that AB is equal to DE . Join HB, ME . Then because the square of AH is equal to the squares of AK, KH , and the squares of AB, BK are equal to the square of AK : the squares of AB, BK, KH will be equal to the square of AH . But the square of BH is equal to the squares of BK, KH , for HKB is a right angle, since HK is perpendicular to the plane of the triangle BAC . Therefore the square of AH is equal to the squares of AB, BH ; and so the angle ABH is a right angle. By the same reason the angle DEM is also a right angle; and the angle BAH is equal to the angle EDM , for so it is made: But AH is equal to DM ; therefore AB is also equal to DE : Therefore because AC is equal to DF , and AB to DE ; and so the two sides CA, AB are equal to the two sides FD, DE , and the angle CAB equal to the angle FDE . Therefore [by 4. 1.] the base BC is equal to the base EF , and one triangle equal to the other, and the rest of the angles equal to the rest of the angles: Wherefore the angle ACB is equal to the angle DFE . But [by constr.] the right angle ACK is equal to the right angle DFN ; and so the remaining angle BCK is equal to the remaining

remaining angle EFN . By the same reason the angle CBK is equal to the angle FEN : Wherefore CBK , FEN are two triangles having two angles of the one equal to two angles of the other, each to each, and one side of the one equal to one side of the other, which is adjacent to the equal angles, *viz.* BC to EF : Therefore [by 26. 1.] the rest of the sides of the one are equal to the rest of the sides of the other: Wherefore CK is equal to FN . But AC is also equal to DF : Wherefore the two sides AC , CK are equal to the two sides DF , FN ; and they contain right angles. Therefore [by 4. 1.] the base AK is equal to the base DN ; and since AH is equal to DM , the square of AH will be equal to the square of DM . But [by 47. 1.] the squares of AK , KH are equal to the square of AH ; for AKH is a right angle; and the squares of DN , NM are equal to the square of DM ; because the angle DNM is a right angle: Therefore the squares of AK , KH are equal to the squares of DN , NM . But the square of AK is equal to the square of DN : Therefore the remaining square of KH is equal to the remaining square of NM ; and so the right line HK is equal to MN . And since the two sides HA , AK are equal to the two sides MD , DN , each to each, and the base HK has been proved to be equal to the base NM ; [by 8. 1.] the angle HAK will be equal to the angle MDN . Which was to be demonstrated.

Corollary. From hence it is manifest, if there be two equal right lined plane angles, and equal right lines be erected from them, containing equal angles with the sides of the first proposed angles, each to each; the perpendiculars drawn from them to the planes wherein are the first proposed angles, are equal to one another.

PROP. XXXVI. THEOR.

If three right lines be proportional, I say the solid parallelepipedon made of all three of them, is equal to the solid parallelepipedon made of the middle line, being equilateral and equiangular to the first mentioned one^w.

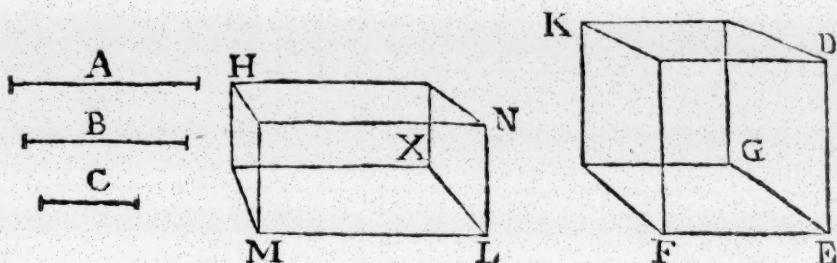
Let the three right lines A , B , C be proportional, as A is to B , so is B to C : I say the solid parallelepipedon made

A a 4

with

with A, B, c is equal to the solid parallelepipedon made from B , equilateral and equiangular to that first mentioned.

Let E be any solid angle contained under the three plane angles DEG, GEF, FED ; make DE, GE, EF each equal to B , and compleat the solid parallelepipedon $E K$. Make LM equal to A , and at the right line LM , and point L in it, make [by 26. II.] the solid angle contained under NLX, XLM, MLN equal to the solid angle



at E . Make LX equal to B , and LN equal to c . Then because A is to B , as B is to c , and A is equal to LM ; and B to either LX, EF, EG , or ED , and c is equal to LN : it will be as LM is to EF , so is DE to LN , and the sides about the equal angles MLN, DEF are reciprocally proportional: Therefore [by 14. 6.] the parallelogram MN is equal to the parallelogram DF . And because the two right lined plane angles DEF, NLM are equal, and the equal right lines LX, EG are elevated from them, containing equal angles with the sides of the angle DEF, NLM , each to each; the perpendiculars drawn from the points G, X to the planes passing thro' NLM, DEF [by cor. 35. II.] are equal to one another: Wherefore the solids HL, EK have the same altitude. But [by 31. II.] solid parallelepipedons that stand upon the same base, and have the same altitude, are equal to one another: Wherefore the solid HL is equal to the solid $E K$. But the solid HL is that made from A, B, c , and the solid $E K$ that made from B . Therefore the solid made from A, B, c , is equal to the solid made from B , being equilateral and equiangular to HL .

If therefore three right lines be proportional, the solid parallelepipedon made of all three of them, is equal to the solid parallelepipedon made of the mean line, being equilateral and equiangular to the first mentioned solid. Which was to be demonstrated.

^w It might be better to say, If three right lines be proportional, the solid parallelepipedon, whose three sides are those three right lines, will be equal to the solid parallelepipedon, whose side is the mean line, being equilateral and equiangular to the solid first mentioned.

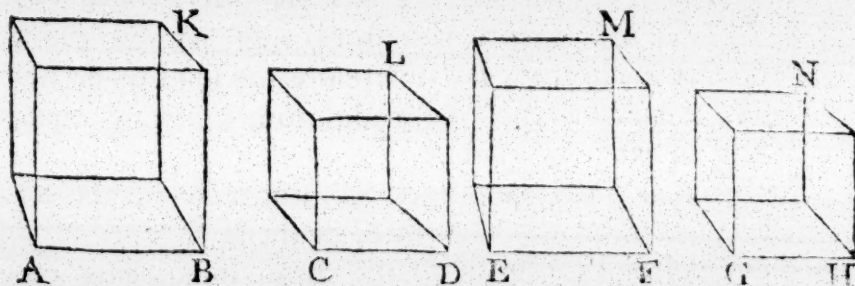
Here it may be observed, that it would have been more simple and natural for Euclid to have called one of the faces of a parallelepipedon, the side of the solid, than to call one of the right lines which is the side of one of the faces, the side of the solid. — If right lines, being the bounds of a superficies, be called their sides, I should think it would be proper to call the superficies, being the bounds of solids, their sides. This is always done in common speech amongst us; for we never call, for instance, the corner of an house its side. But one of its upright faces is always called the side of an house. A parallelepipedon, according to Euclid, has twelve sides.

PROP. XXXVII. THEOR.

If four right lines be proportional, the solid parallelepipedons made from them, being similar and alike described, will be proportional; and if the solid parallelepipedons, similar and alike described, be proportional; those right lines they are described from will be proportional^x.

Let four right lines AB, CD, EF, GH be proportional; and let AB be to CD , as EF is to GH ; and let the similar and alike situate parallelepipedons KA, LC, ME, NG be described from AB, CD, EF, GH : I say as KA is to LC , so is ME to NG .

For because the solid parallelepipedon KA is similar to LC ; KA [by 33. II.] to LC will be in the triplicate ratio of AB to CD : By the same reason the solid ME to the



solid NG will be in the triplicate ratio of EF to GH . But [by

[by supposition] as AB is to CD , so is EF to GH : Therefore as AK is to LC , so is ME to NG .

Now let the solid AK be to the solid LC , as the solid ME is to the solid NG : I say as the right line AB is to the right line CD , so is the right line EF to the right line GH .

For because again AK is to LC in the triplicate ratio of AB to CD , and ME is to NG in the triplicate ratio of EF to GH , and AK is to LC , as ME is to NG ; it will be as AB is to CD , so is EF to GH .

Therefore if four right lines be proportional, the solid parallelepipeds made from them, being similar and alike described, will be proportional; and if the solid parallelepipeds being similar and alike described, be proportional; these right lines upon which they are described, will be proportional. Which was to be demonstrated.

* This proposition is universally true, as is the sixteenth of the sixth book, when the four similar solids are all of the same kind. But when two are of one kind, and two of another, it will not be true, unless the two solids of one kind be described from the first and second, or the first and third, and the two of the other kind from the third and fourth, or second and fourth of the four proportional right lines; the first of these lines being to the second, as the third is to the fourth. See the note upon prop. 16. of the sixth book.

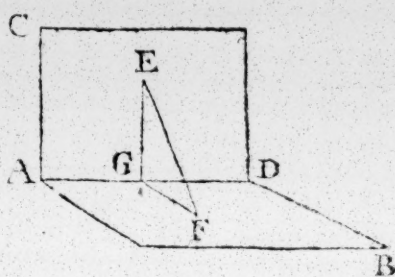
PROP. XXXVIII. THEOR.

If one plane be perpendicular to another; and from any point in one plane, a perpendicular be drawn to the other plane; this will fall in the common section of the planes.

For let the plane CD be perpendicular to the plane AB , and let AD be their common section, and let any point E be taken in the plane CD : I say a perpendicular drawn from E to the plane AB will fall in AD .

For if not, let it, if possible, fall without, as EF meeting the plane AB in the point F : Draw FG [by 10. 1.] from the point F in the plane AB , perpendicular to AD , which [by def. 4. 11.] will be perpendicular to the plane CD ; and join EG . Then because FG is at right angles to the plane CD , and the right line EG , which touches it,

is in the same plane CD ; [by 3. def. 11.] the angle FGE is a right angle. But EF also is at right angles to the plane AB . Wherefore EFG is a right angle: and so two angles of the triangle EFG are right angles; which [by 17. 1.]



is impossible: Wherefore a perpendicular drawn from the point E to the plane AB does not fall out of the right line AD : Therefore it falls in AD .

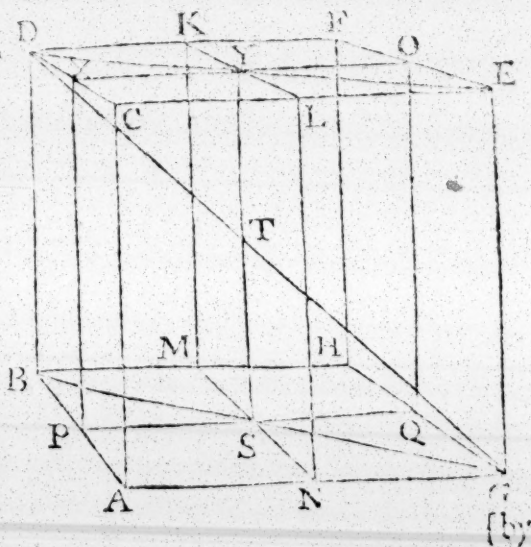
Wherefore if one plane be at right angles to another, a right line drawn from any point in one plane perpendicular to the other, will fall in the common section of the planes. Which was to be demonstrated.

PROP. XXXIX. THEOR.

If the sides of the opposite planes of a solid parallelepipedon be bisected, and planes be drawn thro' the sections; the common section of the planes, and the diameter of the solid parallelepipedon, will mutually bisect each other.

For let the sides of the opposite planes CF , AH of the parallelepipedon DF be bisected in the points K , L , M , N , X , P , O , Q , and thro' the sections draw the planes KX , XQ , and let YS be the common section of the planes, and DC the diameter of the solid parallelepipedon: I say YS , DC do mutually cut one another into two equal parts, that is, YT is equal to TS , and DT equal to TG .

For join DY , YE , BS , SG . Then because DX is parallel to OE ; the alternate angles DXY , YOE [by 29. 1.] are equal to one another; and because DX is equal to OE , and XY to YO , and they contain equal angles; the base DY



[by 4. 1.] will be equal to the base YE , and the triangle DXY is equal to the triangle YOE , and the rest of the angles equal to the rest of the angles: Therefore the angle XYD is equal to the angle OYE ; and so [by 14. 1.] DYE is one right line. By the same reason BSG is one right line, and BS is equal to SG . And because [by 34. 1.] CA is equal and parallel to DB , and CA is also equal and parallel to EG ; DB will be equal to EG , and [by 30. 1.] parallel to it: But the right lines DE , BG join them: Therefore [by 33. 1.] DE is parallel to BG , and in each of them some points D, Y, G, S are taken, and DG, YS are joined: Therefore [by 7. 11.] DG, YS are both in one plane. Wherefore because DE is parallel to BG , the angle EDT [by 29. 1.] will be equal to the angle BGT ; for they are alternate angles. But the angle DTY [by 15. 1.] is equal to the angle GTS : Therefore there are two triangles DTY, GTS , having two angles of the one equal to two angles of the other, each to each, and one side of the one equal to one side of the other, opposite to one of the equal angles, *viz.* DY equal to GS ; for they are the halves of DE, BG : Therefore [by 26. 1.] the rest of the sides will be equal to the rest of the sides; and so DT is equal to TG , and YT equal to TS .

If therefore in a solid parallelepipedon, &c. Which was to be demonstrated.

PROP. XL. THEOR.

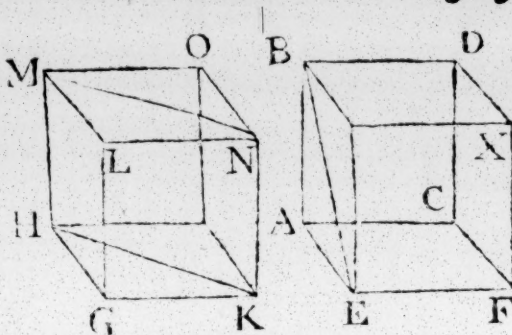
If there be two prisms of the same altitude, and the base of one of them be a parallelogram, and that of the other a triangle, and that parallelogram be double to the triangle; the prisms will be equal.

Let $ABCDEF, GHKLMN$ be prisms of the same altitude. Let the base of one of them be the parallelogram AF , and that of the other the triangle GHK , and let the parallelogram AF be double to the triangle GHK : I say the prism $ABCDEF$ is equal to the prism $GHKLMN$.

For

For complet the solids AX , GO . And because the parallelogram AF is double to the triangle GHK , and [by 34. 1.] the parallelogram HK is double to the triangle GHK ; the parallelogram AF will be equal to the parallelogram HK . But [by 31. 11.] parallelepipeds standing upon equal bases, and having the same altitude, are equal to one another: Therefore the solid AX is equal to the solid GO . But the prism $ABCDEF$ is one half the solid AX , and the prism $GHKLMN$ one half the solid GO : Therefore the prism $ABCDEF$ is equal to the prism $GHKLMN$.

If therefore two prisms, &c. Which was to be demonstrated.



E U C L I D ^s

E L E M E N T S.

B O O K XII.

L E M M A.

If there be two unequal magnitudes proposed, and from the greater be taken away more than one half of it, and again from the remainder be taken away more than its half, and this be always done; there will at last remain some magnitude that will be less than the least of the two given magnitudes.

LET there be two unequal magnitudes AB, C , whereof AB is the greater: I say if from AB be taken away more than one half, and from the remainder be again taken away more than its half, and this be always done, there will at last remain some magnitude less than C .

For if C be multiplied, it will some time or other become greater than the magnitude AB . Let this be done, and let DE be a multiple of C greater than AB . Divide DE into the parts DF, FG, GE , each equal to C , and from AB take away BH more than its half, and again from AH take away HK more than one half of it, and do thus continually until there are the same number of divisions in AB as in DE . Let then these divisions AK, KH, HB , be equal in number to the divisions GF, FG, GE .

And because DE is greater than AB , and there is taken away from DE less than one half of it, viz. EG , and from AB more than its half, viz. BH ; the remainder GD will be greater than the remainder HA . Again, because

GD

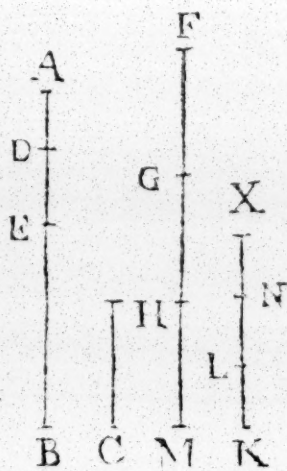
GD is greater than HA , and from GD is taken its half GF , and from HA more than its half HK , the remainder DF will be greater than the remainder AK . But DF is equal to c : Therefore c is greater than AK : Wherefore AK is less than c : Therefore the remainder AK of the magnitude AB , is less than the lesser of the two given magnitudes c . Which was to be demonstrated.

After the same manner we demonstrate the thing, when the halves of the given magnitudes are taken away.

Otherwise.

Let AB, C be two given magnitudes, and let c be the least of them. Then because c is the least, if it be multiplied, it will at length be greater than the magnitude AB . Let this be done, and let FM be the magnitude, which divide into the parts MH, HG, GF , each equal to c ; and take away BE from AB , greater than one half of it; and from EA , take ED greater than one half of it; and doing thus continually until the number of divisions in AB are equal to the number of divisions in FM . Let these be BE, ED, DA ; and KL, LN, NX , each equal to DA ; and do this continually till the divisions of KX be equal in number to the divisions in FM .

Then because BE is greater than one half AB , BE will be greater than EA ; therefore BE is much greater than DA , but xN is equal to DA : wherefore BE is greater xN . Again, because ED is greater than one half EA ; ED will be greater than DA . But NL is equal to DA : wherefore ED is greater than NL : therefore the whole DB is greater than xL : But LK is equal to DA . Therefore the whole AB will be greater than the whole xK . But MF is greater than BA : Wherefore MF is much greater than xK : And because xN, NL, LK are equal to one another,



another, as also MH , HG , GF , and the multitude of them in MF is equal to the multitude of the magnitudes in xK : [by 12. 5.] it will be as KL is to FH , so is xK to FM . But FM is greater than xK : Therefore [by 14. 5.] FG is greater than LK . But FG is equal to c : wherefore KL is equal to AD : therefore c will be greater than AD . Which was to be demonstrated.

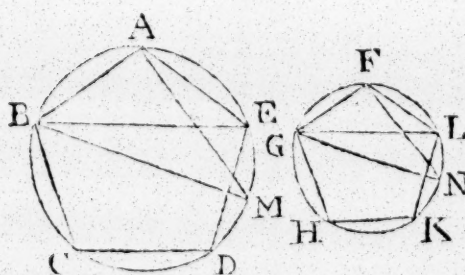
This lemma is the first proposition of the tenth book of Euclid.

PROP. I. THEOR.

Similar polygons inscribed in circles are to one another as the squares of the diameters^a.

Let the circles be $ABCDE$, $FGHKL$, and the similar polygons in them be $ABCDE$, $FGHKL$; and let BM , GN be diameters of the circles: I say as the square of BM is to the square of GN , so is the polygon $ABCDE$ to the polygon $FGHKL$.

For join BE , AM , GL , FN . Then because the polygon $ABCDE$ is similar to the polygon $FGHKL$, and [by def. 1. 6.] the angle BAE is equal to the angle GFL , and as BA is to AE , so is GF to FL : Therefore BAE , GFL are two triangles, having one angle of the one equal to one angle of the other, viz. the angle BAE equal to the angle GFL , and the sides about the equal angles proportional: Wherefore [by 6. 6.] the triangle ABE is equiangular to the triangle FLG ; and accordingly the angle



AEB is equal to the angle FLG . But [by 21. 3.] the angle AEB is equal to the angle AMB ; for they stand upon the same part of the circumference; and the angle FLG is equal to the angle FNG .

And [by 31. 3.] the right angle BAM is equal to the right angle GFN ; and so the remaining angle is equal to the remaining angle: Therefore the triangle ABM is equiangular to the triangle FGN : Therefore [by 4. 6.] as BM is to GN , so is BA to GF . But [by 20. 6.] the ratio of BM to GN is the duplicate of the ratio of the square of BM to the square of GN , and the ratio of the polygon

$ABCDE$

$ABCDE$ to the polygon $FGHKL$ is the duplicate of the ratio of BA to GF : And therefore [by II. 5.] as the square of BM is to the square of GN , so is the polygon $ABCDE$ to the polygon $FGHKL$.

Wherefore similar polygons inscribed in circles are to one another as the squares of the diameters. Which was to be demonstrated.

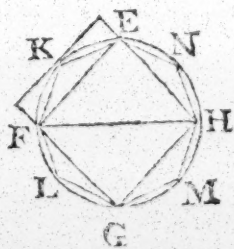
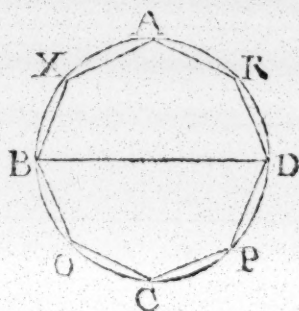
^a The circuits of similar polygons inscribed in circles, are also, as the diameters of the circles.

PROP. II. THEOR.

Circles are to one another as the squares of their diameters^b.

Let the circles be $ABCD$, $EFGH$, and let BD , FH be their diameters. I say as the square of BD is to the square of FH , so is the circle $ABCD$, to the circle $EFGH$.

For if it be not so, it will be as the square of BD is to the square of FH , so is the circle $ABCD$ to some space either greater or less than the circle $EFGH$. First let it be to the space s less than the circle; and in the circle $EFGH$ inscribe a square $EFGH$. Then the square $EFGH$ described in the circle, is greater than one half the circle $EFGH$; because if tangents to the circle be drawn thro' the points E , F , G , H , the square $EFGH$ will be [by 47. I. and 31. 3.] one half the square described about the circle, and the circle is less than the square described about it: Therefore the square $EFGH$ is greater than one half the circle $EFGH$. Bisect the parts of the circumference EF , FG , GH , HE in the points K , L , M , N ; and join EK , KF , FL , LG , GM , MH , HN , NE . Then each of the triangles EKF , FLG , HMG , HNE is greater than one half the segment of the circle wherein it is; because if thro' the points K , L , M , N tangents be drawn to the circle, and



B b

the

the parallelograms upon the right lines EF , FG , GH , HE be completed [by 37. 1.] each of the triangles EKF , FLG , GMH , HNE is one half of its correspondent parallelogram. But the segment is less than the parallelogram; wherefore each of the triangles EKF , FLG , GMH , HNE is greater than one half the segment of the circle wherein it is: Therefore if the rest of the parts of the circumference be bisected, and right lines be joined, and this be always done, there will at last remain some segments of the circle, which will be less than the excess of the circle $EFGH$ above the space s . For it is * demonstrated, in the first theorem of the tenth book, that if there be two proposed unequal magnitudes, and from the greater be taken away more than its half, and from the remainder more than its half, and this be continually done, there will at last remain some magnitude that is less than the least of the proposed magnitudes: Wherefore let the segments of the circle $EFGH$ on the right lines EK , KF , FL , LG , GM , MH , HN , NE , be left, which are greater than the excess whereby the circle $EFGH$ exceeds the space s . Therefore the remaining polygon $EKF LGMHN$ will be greater than the space s . Also in the circle $ABCD$ describe the polygon $AXBOCPDR$ similar to the polygon $EKF LGMHN$: Then [by 1. 12.] as the square of BD is to the square of FH , so is the polygon $AXBOCPDR$ to the polygon $EKF LGMHN$. But [by supposition] as the square of BD is to the square of FH , so is the circle $ABCD$ to the space s : Therefore also as the circle $ABCD$ is to the space s , so is the polygon $AXBOCPDR$ to the polygon $EKF LGMHN$; and inversely, as the circle $ABCD$ is to the polygon within it, so is the space s to the polygon $EKF LGMHN$. But the circle $ABCD$ is greater than the polygon in it: Wherefore the space s is also greater than the polygon $EKF LGMHN$. But [by supposition] it is less too; which is impossible: Therefore it is not as the square of BD is to the square of FH , so is the circle $ABCD$ to some space less than the circle $EFGH$. After the same manner we demonstrate that it is not as the square of FH to the square of BD , so is the circle $EFGH$ to some space less than the circle $ABCD$: I say also it is not as the square of BD to the square of FH ,

* See the lemma at the beginning of this book.

So is the circle $A B C D$ to some space greater than the circle $E F G H$. For if possible, let it be so to a greater space s ; then again inversely it will be as the square of $F H$ is to the square of $B D$, so is the space s to the circle $A B C D$: But [as is demonstrated below] as the space s is to the circle $A B C D$, so is the circle $E F G H$ to some space less than the circle $A B C D$: Therefore as the square of $F H$ is to the square of $B D$, so is the circle $E F G H$ to some space less than the circle $A B C D$; which has been proved to be impossible: Therefore it is not as the square of $B D$ is to the square of $F H$, so is the circle $A B C D$ to some space greater than the circle $E F G H$. But it has been proved that it is not so neither to some space less: Wherefore as the square of $B D$ is to the square of $F H$, so will the circle $A B C D$ be to the circle $E F G H$.

Therefore circles are to one another as the squares of their diameters. Which was to be demonstrated.

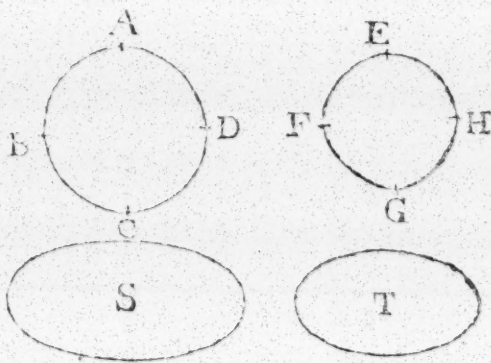
L E M M A.

Now I say, if the space s be greater than the circle $E F G H$, it is as the space s is to the circle $A B C D$, so is the circle $E F G H$ to some space less than the circle $A B C D$.

For make as the space s is to the circle $A B C D$, so is the circle $E F G H$ to the space T . I say the space T is less than the circle $A B C D$. For because it is as the space s to the circle $A B C D$, so is the circle $E F G H$ to the space T ; it will be [by 16. 5.]

as the space s is to the circle $E F G H$, so is the circle $A B C D$ to the space T : But [by supposition] the space s is greater than the circle $E F G H$: Wherefore the circle $A B C D$ is greater than the space T ; and

accordingly as the space s is to the circle $A B C D$, so is the circle $E F G H$ to some space less than the circle $A B C D$.



^b Altho' these sort of exhaustive demonstrations *ad absurdum* be most scrupulously rigorous, and unexceptionably exact, yet their length, and nature, are too apt to give distaste to learners,

ers, and those who have not any great strength of mind, or much inclination to pursue these sort of studies far: they are generally tired and confused before they arrive at the conclusions of these long demonstrations, and oftentimes receive by them but a faint knowledge and slight conviction of the truth of the theorems. It must indeed be confessed, that all sorts of demonstrations *ad absurdum* do not so much prove the truth of the positions, as the absurdity that would follow by granting them to be false; and that there is a much shorter, direct, and easier way, of sufficiently confirming the truth of these sort of theorems by the method of Indivisibles of Bonaventur. Cavillarius, which, tho' liable to some exceptions, by men of penetration and sound logic, yet the idle and the less discerning either see none at all, or overlook them; drowning all lesser faults in the easiness and facility of the method, and smothering every immaterial error, to avoid trouble, tediousness, and confusion.

If circles be conceived to differ from regular polygons, inscribed in them, or circumscribed about them, of exceeding great equal numbers of very small sides, by magnitudes less than any assignable magnitudes; those circles, without any finite error, may be taken for such polygons, and their circumferences for the circumferences of such polygons; and so since all regular polygons of equal numbers of sides, are similar figures; the circles themselves circumscribing or inscribed within such polygons, will be similar figures, equivalent to right lined figures; and accordingly [by 1. 12.] will be to one another as the squares of the diameters; and the circumferences will be as the diameters.

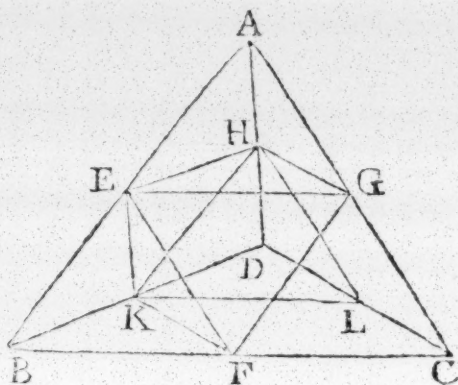
PROP. III. THEOR.

Every pyramid having a triangular base may be divided into two equal and similar pyramids having triangular bases, being similar to the whole; as also into two equal prisms that are greater than one half the whole pyramid.

Let there be a pyramid whose base is the triangle ABC , and vertex the point D . I say the pyramid $ABCD$ may be divided into two pyramids equal and similar to one another, having triangular bases, and similar to the whole, as also into two equal prisms, which two prisms are greater than one half the pyramid.

For

For bisect AB, BC, CA, AD, DB, DC in the points E, F, G, H, K, L , and join $EH, EG, GH, HK, KL, LH, EK, KF, FG$. Then because AE is equal to EB , and AH to HD ; [by 2. 6.] EH will be parallel to DB . By the same reason HK is also parallel to AB : Therefore $HEBK$ is a parallelogram; and so [by 34. 1.] HK is equal to EB . But EB is equal to AE ; and therefore AE will be equal to HK ; but AH is equal to HD : Therefore the two sides AE, AH are equal to the two sides KH, HD , each to each, and [by 29. 1.] the angle EAH is equal to the angle KHD . Where-



fore the base EH [by 4. 1.] is equal to the base KD ; and so the triangle AEH is equal and similar to the triangle HKD . By the same reason the triangle AHG is equal and similar to the triangle $HL D$. And because the two right lines EH, HG touching one another, are parallel to two right lines KD, DL touching one another, and not in the same plane; they will contain equal angles [by 10. 11.]: Therefore the angle EHG is equal to the angle KDL . Again, because the two sides EH, GH are equal to the two sides KD, DL , each to each, and the angle EHG is equal to the angle KDL ; the base EG will be equal to the base KL ; therefore the triangle EHG is equal and similar to the triangle KDL . By the same reason the triangle AEG is equal and similar to the triangle HKL . Wherefore the pyramid whose base is the triangle AEG , and vertex the point H , is equal and similar to the pyramid whose base is the triangle HKL , and vertex the point D . And because HK is drawn parallel to one side AB of the triangle ADB ; [by 29. 1.] the triangle ADB is equiangular to the triangle DHK ; and [by 4. 6.] they have proportional sides: Therefore the triangle ADB is similar to the triangle DHK . And by the same reason the triangle DBC is similar to the triangle DKL , and the triangle ADC to the triangle DHL . And since two right lines BA, AC touching one another, are parallel to two right lines KH, HL touching one another, not be-

ing in the same plane; [by 10. 11.] they will contain equal angles: Therefore the angle BAC is equal to the angle KHL : And it is as BA to AC , so is KH to HL : Therefore [by 6. 6.] the triangle ABC is similar to the triangle HKL ; and so the pyramid, whose base is the triangle ABC , and vertex the point D , is similar to the pyramid, whose base is the triangle HKL , and vertex the point D . But the pyramid, whose base is the triangle HKL , and vertex the point D , has been proved to be equal to the pyramid whose base is the triangle $AE G$, and vertex the point H : And therefore the pyramid, whose base is the triangle ABC ; and vertex the point D , is similar to the pyramid, whose base is the triangle $AE G$, and vertex the point H . Therefore each of the pyramids $AE GH$, $HKL D$ is similar to the whole pyramid $ABCD$. And because BF is equal to FC ; the parallelogram $EBFG$ will be [by 41. 1.] double to the triangle GFC . And because if there be two prisms of the same altitude, one of which has a parallelogram for its base, and the other a triangle, and the parallelogram be double to the triangle; these prisms [by 40. 11.] are equal to one another: Therefore the prisms contained under the two triangles BKF , EHG , and the three parallelograms $EBFG$, $EBKH$, $KHGF$ are equal to the prism contained under the two triangles GFC , HKL , and the three parallelograms $KFCL$, $LCGH$, $HKFG$. And it is manifest that each of the prisms, that whose base is the parallelogram $EBFG$, and opposite to it the right line HK , and that whose base is the triangle GFC , and opposite base the triangle KLH , is greater than either of the pyramids whose bases are the triangles $AE G$, HKL , and vertexes the points H , D . Because if the right lines EF , EK be joined, the prism, whose base is the parallelogram $EBFG$, and opposite to it the right line HK , is greater than the pyramid whose base is the triangle EBF , and vertex the point K . But the pyramid, whose base is the triangle EBF , and vertex the point K , [by 10. def. 11.] is equal to the pyramid, whose base is the triangle $AE G$, and vertex the point H ; for they are contained under equal and similar planes. Therefore the prism, whose base is the parallelogram $EBFG$, and the right line HK opposite to it, is greater than the pyramid, whose base is the triangle $AE G$, and vertex the point H : But the prism
whose

whose base is the parallelogram $E B F G$, and the right line $H K$ is opposite to it, is equal to the prism whose base is the triangle $G F C$, and the triangle $H K L$ is opposite to it; and the pyramid, whose base is the triangle $A E G$, and vertex the point H , is equal to the pyramid, whose base is the triangle $H K L$, and vertex the point D . Therefore the two prisms first mentioned, are greater than the two prisms whose bases are the triangles $A E G$, $H K L$, and vertexes the points H , D . Therefore the whole pyramid, whose base is the triangle $A B C$, and vertex the point D , is divided into two equal pyramids similar to one another and to the whole, and into two equal prisms, which are greater than the whole pyramid. Which was to be demonstrated.

PROP. IV. THEOR.

If there be two prisms of the same altitude, having triangular bases, and either of them be divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms; and if in like manner each of the pyramids made by the former division be divided, and this be continually done: then as the base of one of the pyramids is to the base of the other, so are all the prisms that are in one of the pyramids, to all the prisms in the other pyramid being equal in number.

Let there be two pyramids of the same altitude, having the triangular bases $A B C$, $D E F$, and vertexes the points G , H , and let each of them be divided into two equal pyramids similar to the whole, and into two equal prisms, and let each of the pyramids made by such a division be divided after the same manner; and if this be always done: I say as the base $A B C$ is to the base $D E F$, so are all the prisms in the pyramid $A B C G$, to all the prisms that are in the pyramid $D E F H$, being equal in number.

For because $B X$ is equal to $X C$, and $A L$ to $L C$; [by 2. 6.] $X L$ will be parallel to $A B$, and [by 4. 6.] the triangle $A B C$ is similar to the triangle $L X C$. By the same reason the triangle $D E F$ is similar to the triangle $R Q F$; and because $B C$ is double to $C X$, and $E F$ is double to $F Q$, as $B C$ is to $C X$, so will $E F$ be to $F Q$. And there are

B b 4

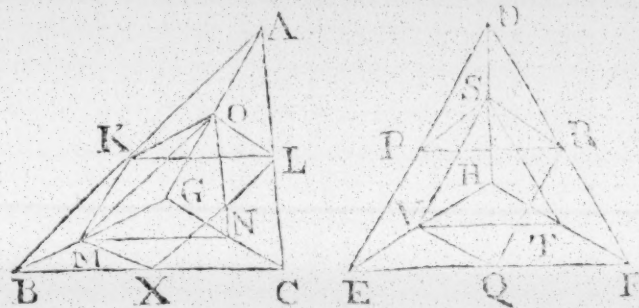
described

described upon BC , CX the similar right lined figures ABC , LXC alike situate, and upon EF , FQ the similar

and alike situate right lined figures

DEF , RQF :

Therefore [by 22. 6.] as the triangle BAC is to the triangle LXC , so is



the triangle DEF to the triangle RQF , and inversely as the triangle ABC is to the triangle DEF , so is the triangle LXC to the triangle RQF . But [as it is demonstrated below] as the triangle LXC is to the triangle RQF , so is the prism whose base is the triangle LXC , and the opposite to it OMN , to the prism whose base is the triangle RQF , and opposite base STV : Therefore [by 11. 5.] as the triangle ABC is to the triangle DEF , so is the prism whose base is the triangle LXC and opposite base OMN , to the prism whose base is the triangle RQF , and opposite base STV . And because the two prisms in the pyramid $ABCG$ are equal to one another, as also the two prisms in the pyramid $DEFH$; the prism whose base is the parallelogram $KLXB$, and the right line MO opposite to it, will be to the prism whose base is the triangle LXC , and opposite base OMN , as the prism whose base is the parallelogram $EPRQ$, and opposite to it the right line SV , to the prism whose base is the triangle RQF , and opposite base is STV . Wherefore by composition [by 18. 5.] as the prisms $KBLXMO$, $LXCMNO$, to the prism $LXCMNO$, so are the prisms $PEQRST$, $RQFSTV$ to the prism $RQFSTV$: And by alternation, as the prisms $KBLXMO$, $LXCMNO$ to the prisms $PEQRST$, $RQFSTV$, so is the prism $LXCMNO$, to the prism $RQFSTV$. But it has been proved that as the prism $LXCMNO$ is to the prism $RQFSTV$, so is the base LXC to the base RQF , and the base ABC to the base DEF : Therefore as the triangle ABC is to the triangle DEF , so are the two prisms that are in the pyramid $ABCG$, to the two prisms in the pyramid $DEFH$. So likewise if the pyramids made by the division aforesaid be divided after the same manner as $OMNO$, $STVH$, it will be as the base

base OMN to the base STV , so are the two prisms in the pyramid $OMNG$ to the two prisms that are in the pyramid $STVH$. But as the base OMN is to the base STV , so is the base ABC to the base DEF . And therefore as the base ABC is to the base DEF , so are the two prisms in the pyramid $ABCG$ to the two prisms that are in the pyramid $DEFH$, and the two prisms that are in the pyramid $OMNG$ to the two prisms in the pyramid $STVH$, and so are four to four. And the same thing is also proved in the other prisms made by the divisions of the pyramids $AKLO$, and $DPRS$, and of all the rest equal in number. Which was to be demonstrated.

L E M M A.

But we thus demonstrate that the triangle LXC is to the triangle RQF , as the prism whose base is the triangle LXC , and opposite base OMN , to the prism whose base is the triangle RQF , and opposite base the triangle STV .

For in the same figure conceive perpendiculars to be drawn from the points G, H , to the planes of the triangles ABC, DEF , which will be equal to one another; because the pyramids themselves are supposed to have the same altitude, and because the two right lines, viz. GC , and the perpendicular drawn from the point G , are cut by the parallel planes ABC, OMN ; [by 17. 11.] they will be cut in the same ratio. But GC is bisected by the plane OMN in the point N : Therefore the perpendicular drawn from the point G to the plane ABC will be bisected by the plane OMN . By the same reason the perpendicular drawn from the point H to the plane DEF , will be bisected by the plane STV . But the perpendiculars drawn from the points G, H to the planes ABC, DEF are equal: And therefore the perpendiculars drawn from the triangles OMN, STV , to ABC, DEF , are equal: Wherefore the prisms whose bases are the triangles LXC, RQF , and opposite bases the triangles OMN, STV , have the same altitude: Wherefore [by 32. 11.] the solid parallelepipeds which are described upon the said prisms of equal altitudes, are to one another as their bases: and so are their halves: Therefore as the base LXC is to the base RQF , so will the said prisms be to one another. Which was to be demonstrated.

PROP. V. THEOR.

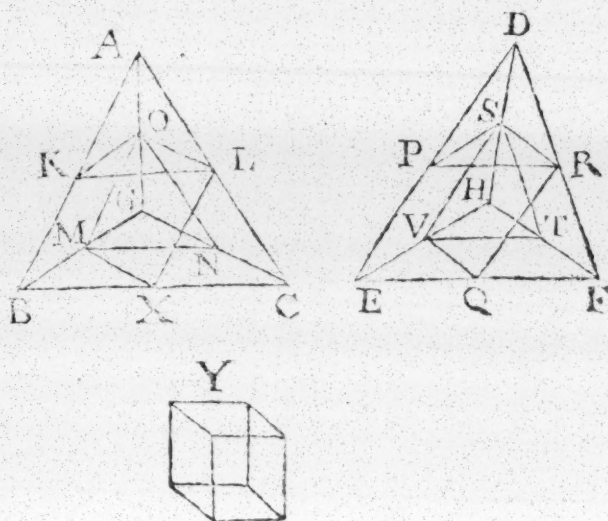
Pyramids of the same altitude, which have triangular bases, are to one another as their bases.

For let pyramids of the same altitude have the triangles ABC , DEF , for their bases, and vertexes the points G , H . I say as the base ABC is to the base DEF , so is the pyramid $ABCG$ to the pyramid $DEFH$.

For if it be not so, it will be as the base ABC is to the base DEF , so is the pyramid $ABCG$ to some solid either less or greater than the pyramid $DEFH$. First let it be to a solid less, as Y ; and divide the pyramid $DEFH$ into two equal pyramids similar to the whole, and into two equal prisms: Then [by 3. 12.] the prisms both together are greater than half the whole pyramid. And again, let the pyramids made by the divisions aforesaid be in like manner divided; and let this be always done, till some pyramids be obtained less than the excess of the pyramid $DEFH$ above the solid Y . Let these pyramids be taken, and let them, for example, be the pyramids $DP RS$, $ST VH$: Therefore the rest of the prisms in the pyramid $DEFH$ will be greater than the solid Y . Also divide the pyramid $ABCG$ after the like manner, into the same number of parts as the pyramid $DEFH$: Therefore [by 4. 12.] as the base ABC is to the base DEF , so are the prisms in the pyramid $ABCG$ to the prisms in the pyramid $DEFH$. But [by supposition] as the base ABC is to the base DEF , so is the pyramid $ABCG$ to the solid Y : And therefore as the pyramid $ABCG$ is to the solid Y , so are the prisms in the pyramid $ABCG$ to the prisms in the pyramid $DEFH$: And alternately as the pyramid $ABCG$ is to the prisms that are in it, so is the solid Y to the prisms that are in the pyramid $DEFH$. But the pyramid $ABCG$ is greater than all the prisms that are in it; and therefore the solid Y is greater than all the prisms in the pyramid $DEFH$: But it is less too; which is impossible. Therefore it is not as the base ABC is to the base DEF , so is the pyramid $ABCG$ to some solid less than the pyramid $DEFH$. In like manner we demonstrate that it is not as the base DEF to the base ABC , so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$: And moreover I say that it

is not as the base ABC to the base DEF , so is the pyramid $ABCG$ to some solid which is greater than the pyramid $DEFH$. For if possible, let it be so to some greater solid as Y . Then inversely it will be as the base DEF is to the base ABC , so is the solid Y to the pyramid $ABCG$.

But as the solid Y is to the pyramid $ABCG$, so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$, as has been but now demonstrated: And therefore as the base DEF , is to the base ABC , so is the pyramid $DEFH$ to some



solid less than the pyramid $ABCG$; which is absurd. Therefore it is not as the base ABC to the base DEF , so is the pyramid $ABCG$ to some solid greater than the pyramid $DEFH$: But it has also been demonstrated not to be so to any solid less: Wherefore as the base ABC is to the base DEF , so is the pyramid $ABCG$, to the pyramid $DEFH$.

Therefore pyramids of the same altitude, having triangular bases, are to one another as their bases. Which was to be demonstrated.

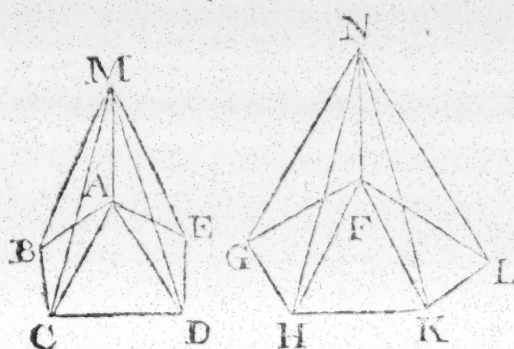
PROP. VI. THEOR.

Pyramids of the same altitude, and having polygonous bases, are to one another as their bases.

Let there be pyramids of the same altitude, whose polygonous bases are $ABCDE$, $FGHKL$, and vertexes the points M , N : I say as the base $ABCDE$ is to the base $FGHKL$, so is the pyramid $ABCDEM$ to the pyramid $FGHKLN$.

For divide the base $ABCDE$ into the triangles ABC , ACD , ADE , and the base $FGHKL$ into the triangles FGH ,

FGH , FHK , FKL , and upon every one of these triangles conceive pyramids to stand of the same altitude with the first proposed pyramids. Then because [by 5. 12.] as the triangle ABC is to the triangle ACD , so is the pyramid $ABCM$, to the pyramid $ACDM$; and by composition [by 18. 5.] as the trapezium $ABCD$ is to the triangle ACD , so is the pyramid $ABCDM$ to the pyramid $ABCM$. But as the triangle ACD is to the triangle ADE , so is the pyramid $ACDM$ to the pyramid $ADEM$. Therefore by equality of ratio [by 22. 5.] as the base $ABCD$ is to the base ADE , so is the pyramid $ABCDM$ to the pyramid $ADEM$. And again by composition, as the base



$ABCDE$ is to the base ADE , so is the pyramid $ABCDEM$ to the pyramid $ADEM$. By the same reason also as the base $FGHLK$ is to the base FKL , so is the pyramid $FGHLN$ to the pyramid $FKLN$.

And because $ADEM$,

$FKLN$ are two pyramids of the same altitude having triangular bases; the base ADE will be to the base FKL , as the pyramid $ADEM$ is to the pyramid $FKLN$: Therefore because it is as the base $ABCDE$ is to the base ADE , so is the pyramid $ABCDEM$ to the pyramid $ADEM$; and as the base ADE is to the base FKL , so is the pyramid $ADEM$ to the pyramid $FKLN$: It will be [by equality of ratio] as the base $ABCDE$ is to the base FKL , so is the pyramid $ABCDEM$ to the pyramid $FKLN$. But also as the base FKL is to the base $FGHLK$, so was the pyramid $FKLN$ to the pyramid $FGHLN$: Wherefore again, by equality of ratio, as the base $ABCDE$ is to the base $FGHLK$, so is the pyramid $ABCDEM$ to the pyramid $FGHLN$.

Therefore pyramids of the same altitude, having polygonous bases, are to one another as their bases. Which was to be demonstrated.

^c All pyramids of the same altitude, are to one another, as their bases, which some easily demonstrate by the method of indivisibles; for since any pyramid may be conceived to be made

made up of an infinite number of little solids, all of the same altitude, having all their bases similar and parallel to the base of the pyramid, perpetually decreasing to its vertex: and since every one of these little solids, being of the same altitude, is as its base: Therefore all such little solids in one pyramid, to all such solids in another of the same altitude, that is, one pyramid will be to the other, as the base of one of the little solids in one pyramid, cut off by a plane parallel to the plane of the base, is to the correspondent base of the other little solid in the other pyramid made by that cutting plane; that is, because the correspondent bases of these little solids are proportional to the bases of the pyramids, the pyramids themselves will be as their bases.

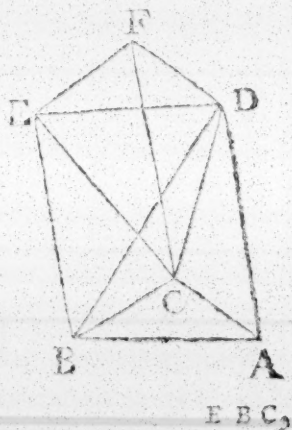
Hence if there be two solids of the same altitude, and if they be both cut any where by a plane parallel to the plane of their bases; and the ratio of the sections made be constant, those solids will be in that constant ratio; that is, in the ratio of the base of the one, to the base of the other; so that, if those sections are equal, and thence the bases, the solids themselves will be equal.

P R O P. VII. T H E O R.

Every prism having a triangular base, is divisible into three equal pyramids having triangular bases.

Let there be a prism whose base is the triangle $A B C$, and opposite base the triangle $D E F$: I say the prism $A B C D E F$ is divisible into three equal pyramids having triangular bases.

For join $B D$, $E C$, $C D$. Then because $A B E D$ is a parallelogram whose diameter is $B D$; the triangle $A B D$ [by 34. 1.] will be equal to the triangle $E D B$. Therefore [by 5. 12.] the pyramid whose base is the triangle $A B D$, and vertex the point C , is equal to the pyramid whose base is the triangle $E D B$, and vertex the point C : But the pyramid whose base is the triangle $E D B$, and vertex the point C , is the same as the pyramid whose base is the triangle $E B C$, and vertex the point D ; for it is contained under the same planes: And therefore the pyramid whose base is the triangle $A B D$ and vertex the point C , is equal to the pyramid whose base is the triangle



EBC , and vertex the point D . Again, because $FCBE$ is a parallelogram whose diameter is CE , the triangle ECF [by 34. 1.] is equal to the triangle CBE : Therefore [by 5. 12.] the pyramid whose base is the triangle BEC , and vertex the point D , is equal to the pyramid whose base is the triangle ECF , and vertex the point D . But the pyramid whose base is the triangle BCE , and vertex the point D , has been proved to be equal to the pyramid whose base is the triangle ABD , and vertex the point C . Wherefore the pyramid whose base is the triangle CEF , and vertex the point D is equal to the pyramid whose base is the triangle ABD , and vertex the point C . Therefore the prism $ABCDEF$ is divided into three equal pyramids, having triangular bases. And because the pyramid whose base is the triangle ABD , and vertex the point C , is the same as the pyramid whose base is the triangle CAB , and vertex the point D ; for it is contained under the same planes; and the pyramid whose base is the triangle ABD , and vertex the point C , has been proved to be a third part of the prism, whose base is the triangle ABC , and the triangle DEF opposite to it: Therefore the pyramid whose base is the triangle ABC , and vertex the point D , is a third part of the prism having the same triangular base ABC , the opposite base being the triangle DEF . Which was to be demonstrated.

Corollary. From hence it is manifest, that every pyramid is a third part of a prism having the same base and altitude. Because if the base of a prism is any other right lined figure, and another such a one, opposite to it, it is divisible into prisms having triangular bases, and the opposite bases also triangles.

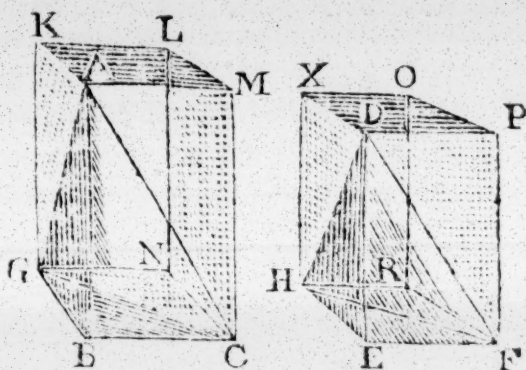
PROP. VIII. THEOR.

Similar pyramids having triangular bases, are in the triplicate ratio of their homologous sides.

Let there be similar and alike-situate pyramids, having the triangular bases ABC , DEF , and vertexes the points G , H . I say the pyramid $ABCG$ to the pyramid $DEFH$ is in the triplicate ratio of the side BC to the side EF .

For compleat the solid parallelepipeds $BGML$, $EHPO$. Then because the pyramid $ABCG$ is similar to the pyramid

mid $DEFH$, the angle ABC [by 9. def. 11.] will be equal to the angle DEF , the angle GBC to the angle HEF , and the angle ABG to the angle DEH , and AB is to DE , as BC is to EF , and as BG is to EH . Therefore because it is as AB is to DE , so is BC to EF , and the sides about the equal angles are proportional: the parallelogram BM



will be similar to the parallelogram EP . By the same reason the parallelogram BN is similar to the parallelogram ER , and the parallelogram BK to the parallelogram EX : Therefore the three parallelograms EM , KB , BN are similar to the three parallelograms EP , EX , ER . But [by 24. 11.] the three parallelograms MB , BK , BN are similar and equal to the three opposite parallelograms, and the three parallelograms EP , EX , ER similar and equal to the three opposite ones; wherefore the solids $BGMN$, $EHPO$ are contained under equal numbers of similar planes; and so [by 9. def. 11.] the solid $BGMN$ is similar to the solid $EHPO$. But [by 33. 11.] similar solid parallelepipeds are in the triplicate ratio of their homologous sides: Therefore the solid $BGMN$ to the solid $EHPO$ is in the triplicate ratio of the homologous side BC to the homologous side EF . But [by 15. 5.] as the solid $BGMN$ is to the solid $EHPO$, so is the pyramid $ABCG$ to the pyramid $DEFH$; for the pyramid is a sixth part of that solid, since the prism, which is one half of the solid parallelepipedon is thrice the pyramid: Therefore the pyramid $ABCG$ to the pyramid $DEFH$ is in the triplicate ratio of BC to EF . Which was to be demonstrated.

Corollary. From hence it is manifest, that similar pyramids having polygonous bases, are to one another in the triplicate ratio of the homologous sides: For they being divided into pyramids having triangular bases, because [by 20. 6.] similar polygonous bases are divided into equal numbers of similar triangles homologous to the wholes; it will be as one of the pyramids having a triangular base in one pyramid, to one pyramid having a triangular base in the

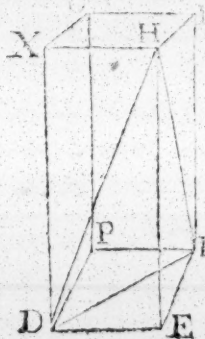
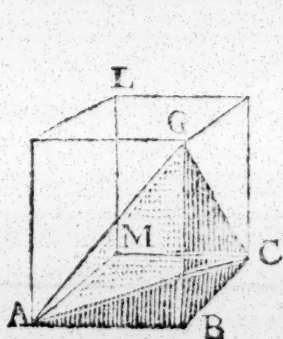
the other, so are all the pyramids having triangular bases in the one pyramid, to all those having triangular bases in the other; that is, so is one pyramid having the polygonous base to the other pyramid having the polygonous base. But one pyramid with a triangular base is to another pyramid with a triangular base in the triplicate ratio of their homologous sides: And therefore one pyramid having a polygonous base to a similar pyramid having such a base, is in the triplicate ratio of one of its homologous sides to the other.

PROP. IX. THEOR.

The bases of equal pyramids, having triangular bases, are reciprocally proportional to their altitudes, and those triangular pyramids whose bases are reciprocally proportional to their altitudes, are equal to one another.

For let there be equal pyramids having the triangular bases ABC , DEF , and vertexes the points G , H : I say the bases of the pyramids ABC , DEF are reciprocally proportional to their altitudes; that is, as the base ABC is to the base DEF , so is the altitude of the pyramid DEF to the altitude of the pyramid ABC .

For compleat the solid parallelepipeds $BGML$, $EHPO$. Then because the pyramid ABC is equal to the pyramid DEF , and the pyramid ABC is the sixth part of the solid $BGML$, and the pyramid DEF the sixth part of the solid $EHPO$: The solid $BGML$ [by 15. 5.] will be equal to the solid $EHPO$. But [by 34. 11.] the bases of equal solid parallelepipeds are reciprocally proportional to their altitudes: Therefore as the



base BM is to the base EP , so is the altitude of the solid $EHPO$ to the altitude of the solid $BGML$. But as the base BM is to the base EP , so is the triangle ABC to the triangle DEF : Therefore as the triangle

gle

gle ABC is to the triangle DEF , so is the altitude of the solid $EHPO$ to the altitude of the solid $BGML$. But the altitude of the solid $EHPO$ is the same as the altitude of the pyramid $DEFH$, and the altitude of the solid $BGML$ the same as the altitude of the pyramid $ABCG$: Therefore as the base ABC is to the base DEF , so is the altitude of the pyramid $DEFH$ to the altitude of the pyramid $ABCG$: Wherefore the bases of the pyramids $ABCG$, $DEFH$ are reciprocally proportional to their altitudes.

Now let the bases of the pyramids $ABCG$, $DEFH$ be reciprocally proportional to their altitudes, and let the base ABC be to the base DEF , as the altitude of the pyramid $DEFH$ is to the altitude of the pyramid $ABCG$: I say the pyramid $ABCG$ is equal to the pyramid $DEFH$.

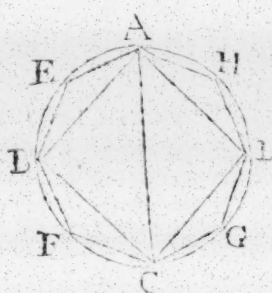
For the same construction remaining, because as the base ABC is to the base DEF , so is the altitude of the pyramid $DEFH$ to the altitude of the pyramid $ABCG$. And as the base ABC is to the base DEF , so is the parallelogram BM to the parallelogram EP ; it will be as the parallelogram BM to the parallelogram EP , so is the altitude of the pyramid $DEFH$ to the altitude of the pyramid $ABCG$. But the altitude of the pyramid $DEFH$ is the same as the altitude of the solid parallelepipedon $EHPO$; and the altitude of the pyramid $ABCG$ the same as the altitude of the solid parallelepipedon $BGML$: Therefore as the base BM is to the base EP , so is the altitude of the solid parallelepipedon $EHPO$ to the altitude of the solid parallelepipedon $BGML$. But those solid parallelepipeds whose bases are reciprocally proportional to their altitudes [by 34. 11.] are equal to one another. Therefore the solid parallelepipedon $BGML$ is equal to the solid parallelepipedon $EHPO$. And the pyramid $ABCG$ is a sixth part of the solid $BGML$, and the pyramid $DEFH$ is also a sixth part of the solid $EHPO$; therefore the pyramid $ABCG$ is equal to the pyramid $DEFH$.

Wherefore the bases of equal pyramids having triangular bases, are reciprocally proportional to their altitudes; and those triangular pyramids, whose bases are reciprocally proportional to their altitudes, are equal to one another. Which was to be demonstrated.

PROP. X. THEOR.

Every cone is a third part of a cylinder which has the same base and an equal altitude^d.

For let a cone have the same base as a cylinder, *viz.* the circle $ABCD$, and an equal altitude: I say the cone is a third part of the cylinder; that is, the cylinder is thrice the cone. For if the cylinder



be not thrice the cone; it will be either greater than thrice the cone, or less. First let it be greater than thrice the cone; and describe a square $ABCD$ in the circle $ABCD$: Then the square $ABCD$ is greater than one half the circle $ABCD$. And upon the square

$ABCD$ erect a prism of the same altitude as the cylinder, which prism will be greater than one half the cylinder; because if a square be described about the circle $ABCD$, the inscribed square will be one half the circumscribed square; and there are erected upon those bases solid parallelepipeds of the same altitude, *viz.* the prisms themselves; and so the prisms are to one another as their bases; therefore the prism erected upon the square $ABCD$ is one half the prism erected upon the square described about the circle $ABCD$; and the cylinder is less than the prism erected upon the square described about the circle $ABCD$: Therefore the prism described upon the square $ABCD$ of the same altitude as the cylinder, is greater than one half that cylinder. Bisect the parts AB, BC, CD, DA of the circumference in the points E, F, G, H , and join $AE, EB, BF, FC, CG, GD, DH, HA$. Then each of the triangles AEB, BFC, CGD, DHA is greater than one half the segment of the circle $ABCD$ wherein it stands, as has been already proved [see 2. 12.]. Upon each of the triangles AEB, BFC, CGD, DHA erect a prism of the same altitude as the cylinder: Then will each of these prisms be greater than one half the respective segment of the cylinder; because if thro' the points E, F, G, H parallels be drawn to AB, BC, CD, DA , and parallelograms at them

be completed, upon which solid parallelepipeds of the same altitude as the cylinder are erected; each of these erected parallelepipeds will be double the prisms which are in the triangles AEB , BFC , CGD , DHA ; and the segments of the cylinder are less than the erected solid parallelepipeds; therefore the prisms in the triangles AEB , BFC , CGD , DHA are greater than one half the respective segments of the cylinder; and so let the remaining parts of the circumference be bisected, right lines be joined, and upon each of the triangles erect prisms of the same altitude as the cylinder, and do this always, till at last [by 1. 10.] some segments of the cylinder remain being less than the excess of the cylinder above thrice the cone. Let such segments be left, and let them be AE , EB , BF , FC , CG , GD , DH , HA . Then the remaining prism, whose base is the polygon $AEBFCGDH$, and altitude the same as that of the cylinder, is greater than thrice the cone. But [by cor. 7. 12.] the prism whose base is the polygon $AEBFCGDH$, and altitude the same as that of the cylinder, is thrice the pyramid whose base is the polygon $AEBFCGDH$, and vertex the same as that of the cone; and therefore the pyramid whose base is the polygon $AEBFCGDH$, and vertex the same as that of the cone, is greater than the cone whose base is the circle $ABCD$: but it is less too, because it is contained in it, which is impossible: Therefore the cylinder is not greater than thrice the cone. I say moreover, that the cylinder is not less than thrice the cone. For if possible let the cylinder be less than thrice the cone; then inversely the cone will be greater than a third part of the cylinder. Describe the square $ABCD$ in the circle $ABCD$; this square will be greater than one half the circle $ABCD$, and upon the square $ABCD$ erect a pyramid, having the same vertex as that of the cone; then the erected pyramid will be greater than one half the cone; because, as we have already demonstrated, if a square be described about the circle, the square $ABCD$ will be one half of it. And if solid parallelepipeds be erected upon those squares, having the same altitude with that of the cone, which are also called prisms; that erected upon the square $ABCD$ will be one half of that erected upon the square described about the

circle; for [by 32. 11.] they are to one another as their bases; and so also are the third parts of them: Wherefore the pyramid whose base is the square $ABCD$ is one half of the pyramid erected upon the square described about the circle. But the pyramid erected upon the square described about the circle, is greater than the cone, for this is contained in that. Therefore the pyramid whose base is the square $ABCD$, and vertex the same as that of the cone, is greater than one half the cone. Bisect the parts AB , BC , CD , DA of the circumference in the points E , F , G , H , and join AE , EB , BF , FC , CG , GD , DH , HA ; then each of the triangles AEB , BFC , CGD , DHA is greater than one half of the respective segment of the circle $ABCD$: Upon each of the triangles AEB , BFC , CGD , DHA erect pyramids of the same altitude as that of the cone; then every one of these pyramids thus erected, is greater than one half the respective segment of the cone: And so bisecting the remaining parts of the circumference, and joining right lines, and erecting a pyramid upon every one of the triangles having the same vertex as that of the cone, and doing this continually, there will at last [by 1. 10.] be left some portions of the cone that will be less than the excess of the cone above one third part of the cylinder. Let there be such left, which let be those upon AE , EB , BF , FC , CG , GD , DH , HA . Then the remaining pyramid, whose base is the polygon $AEBFCGDH$, and vertex the same as that of the cone, is greater than a third part of the cylinder. But [by cor. 7. 12.] the pyramid whose base is the polygon $AEBFCGDH$, and vertex the same as that of the cone, is one third part of the prism whose base is the polygon $AEBFCGDH$, and altitude the same as that of the cylinder: Therefore the prism whose base is the polygon $AEBFCGDH$, and altitude the same as that of the cylinder, is greater than the cylinder whose base is the circle $ABCD$. But it is also less: For it is comprehended by it; which is impossible. Therefore the cylinder is not less than thrice the cone. It has also been proved not to be greater than thrice the cone: Therefore the cylinder is thrice the cone; and accordingly the cone is one third part of the cylinder.

Therefore

Therefore every cone is one third part of a cylinder having the same base as it, and an equal altitude. Which was to be demonstrated.

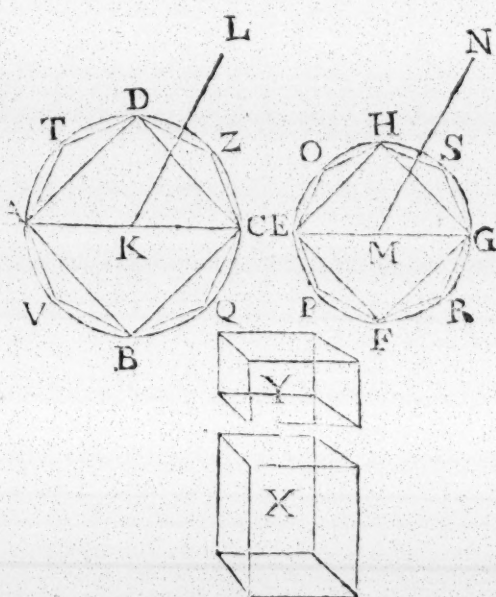
^d Some demonstrate this theorem thus: Every cone may be considered as a pyramid with a polygonous base, of an infinite number of exceeding small equal sides; and every cylinder as a prism with such a polygonous base. And therefore since [by cor. prop. 7.] every pyramid is a third part of a prism of the same base and altitude, every cone, thus taken for a pyramid, will also be one third of a cylinder, of the same altitude, thus taken for a prism.

PROP. XI. THEOR.

Cones and cylinders having the same altitude, are to one another as their bases^e.

Let the cones and cylinders whose bases are the circles $ABCD$, $EFGH$, axes KL , MN , and diameters of the bases are AC , EG , have the same altitude: I say as the circle $ABCD$ is to the circle $EFGH$, so is the cone AL , to the cone EN .

For if it be not so, it will be as the circle $ABCD$ to the circle $EFGH$, so is the cone AL to some solid either less or greater than the cone EN . First let it be so to the solid x , which is less. Let the solid y be equal to the excess of the cone EN above the solid x . Then the cone EN is equal to both the solids x , y . In the circle $EFGH$ describe the square $EFGH$: This square is greater than one half the circle. Upon the square $EFGH$ erect a pyramid of the same altitude as the cone: Then this pyramid is greater than one half the cone: for if a square be described about the circle, and upon it be erected a pyramid of the same altitude



as the cone, the pyramid upon the inscribed square is one half that upon the circumscribing square; for [by 6. 12.] they are to one another as their bases; and the cone is less than the circumscribed pyramid: Therefore the pyramid whose base is the square $EFGH$, and vertex the same as that of the cone, is greater than one half the cone. Bisect the parts EF , FG , GH , HE of the circumference in the points O , P , R , S ; and join HO , OE , EP , PF , FR , RG , GS , SH . Then each of the triangles HOE , EPF , FRG , GSH is greater than one half of the respective segment of the circle. Upon every one of the triangles HOE , EPF , FRG , GSH erect a pyramid of the same altitude with that of the cone. Then each of these erected pyramids is greater than one half the respective segment of the cone; and so bisecting the remaining parts of the circumference, joining right lines, and upon every one of the triangles erecting pyramids of the same altitude as that of the cone, and always doing this, there will at last [by 1. 10.] be left some segments of the cone that will be less than the solid Y . Let such be left; and let them be those that stand on HO , OE , EP , PF , FR , RG , GS , SH , and the remaining pyramid, whose base is the polygon $HOEPFRGS$, and altitude the same as that of the cone, is greater than the solid X . In the circle $ABCD$ describe the polygon $DTAVBQCZ$, similar and alike situate to the polygon $HOEPFRGS$, and upon it raise a pyramid of the same altitude as the cone AL . Then because [by 20. 6. and 1. 12.] as the square of AC is to the square of EG , so is the polygon $DTAVBQCZ$ to the polygon $HOEPFRGS$, and as the square of AC is to the square of GH , so is [by 2. 12.] the circle $ABCD$ to the circle $EFGH$; [by 11. 5.] as the circle $ABCD$ is to the circle $EFGH$, so is the polygon $DTAVBQCZ$ to the polygon $HOEPFRGS$. But [by constr.] as the circle $ABCD$ is to the circle $EFGH$, so is the cone AL to the solid X ; and [by 6. 12.] as the polygon $DTAVBQCZ$ is to the polygon $HOEPFRGS$, so is the pyramid whose base is the polygon $DTAVBQCZ$ and vertex the point L , to the pyramid whose base is the polygon $HOEPFRGS$, and vertex the point N : Therefore as the cone AL is to the solid X , so is the pyramid whose base is the polygon $DTAVBQCZ$, and vertex the point L , to the pyramid whose base is the polygon

$HOEPFRGS$,

HOEPFRGS, and vertex the point N: Wherefore alternately as the cone AL is to the pyramid which is in it, so is the solid x to the pyramid which is in the cone EN. But the cone AL is greater than the pyramid which is in it: Therefore the solid x is greater than the pyramid which is in the cone EN: But it is less too; which is absurd. Therefore it is not as the circle ABCD to the circle EFGH, so is the cone AL to some solid less than the cone EN. In like manner we demonstrate that it is not as the circle EFGH to the circle ABCD, so is the cone EN to some solid less than the cone AL: I say also that it is not as the circle ABCD to the circle EFGH, so is the cone AL to some solid greater than the cone EN. For if possible, let it be so to some solid x, which is greater. Then inversely, as the circle EFGH is to the circle ABCD, so will the solid x be to the cone AL. But as the solid x is to the cone AL, so is the cone EN to some solid less than the cone AL: And therefore as the circle EFGH is to the circle ABCD, so is the cone EN to some solid less than the cone AL; which has been demonstrated to be impossible; therefore it is not as the circle ABCD is to the circle EFGH, so is the cone AL to some solid greater than the cone EN. It has also been proved that it is not so to any solid which is less. Therefore as the circle ABCD is to the circle EFGH, so is the cone AL to the cone EN. But as cone is to cone, so is cylinder to cylinder; for [by 10. 12.] the one is thrice the other; and therefore as the circle ABCD is to the circle EFGH, so are the cones and cylinders of the same altitude standing upon them.

Wherefore cones and cylinders of the same altitude, are to one another as their bases. Which was to be demonstrated.

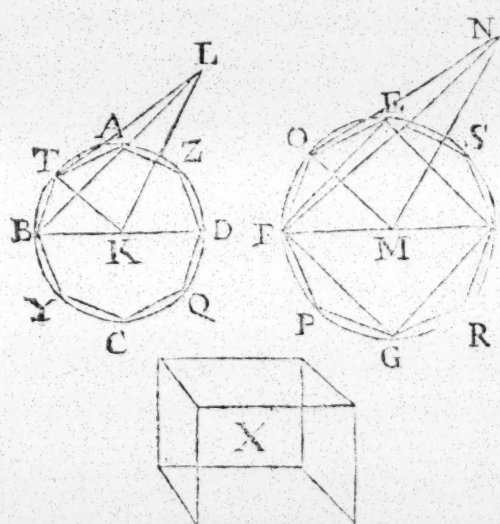
* This may be demonstrated shorter, by the method of indivisibles, and from the notes upon the sixth and tenth theorems.

PROP. XII. THEOR.

Similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases^t.

Let there be similar cones and cylinders, whose bases are the circles $ABCD$, $EFGH$, diameters of the bases BD , FH , and axes of the cones or cylinders KL , MN : I say the cone whose base is the circle $ABCD$, and vertex the point L , will be to the cone whose base is the circle $EFGH$, and vertex the point N , in the triplicate ratio of BD to FH .

For if the cone $ABCDL$ be not to the cone $EFGHN$ in the triplicate ratio of BD to FH , the cone $ABCDL$ will have a triplicate ratio to some solid either less than the cone



$EFGHN$, or greater. First let it have such a ratio to the solid x which is less; and in the circle $EFGH$ describe the square $EFGH$. Therefore the square $EFGH$ is greater than one half the circle $EFGH$. Again, upon the square $EFGH$ erect a pyramid of the same altitude as that of the cone: and so this

erected pyramid is greater than one half the cone. Divide the parts EF , FG , GH , HE of the circumference into equal parts in the points O , P , R , S , and join EO , OF , FP , PG , GR , RH , HS , SE : Then each of the triangles EOF , FPG , GRH , OSE is greater than one half of the respective segment of the circle $EFGH$. Again, upon each of the triangles EOF , FPG , GRH , HSE erect a pyramid, having the same vertex as the cone, and each of the erected pyramids is greater than one half of the respective segment of the cone. Therefore bisecting the remaining segments of the circumference, and joining right lines, and erecting pyramids upon every one of the triangles having the same vertex as the cone, and always doing this, at last

[by

[by 1. 10.] there will be left some segments of the cone which will be less than the excess of the cone $EFGHN$ above the solid x . Let such be left, and let them be those described upon $EO, OF, FP, PG, GR, RH, HS, SE$: Then the remaining pyramid whose base is the polygon $EOFPGRHS$, and vertex the point N , is greater than the solid x . Also in the circle $ABCD$ describe the polygon $ATBV CQDZ$ similar and alike situate to the polygon $EOFPGRHS$; and upon it erect a pyramid, having the same vertex as that of the cone; and let $LB T$ be one of the triangles containing the pyramid, whose base is the polygon $ATBV CQDZ$, and vertex the point L , and NFO one of the triangles containing the pyramid, whose base is the polygon $EOFPGRHS$, and vertex the point N . And lastly join KT, MO . Then because the cone $ABCDL$ is similar to the cone $EFGHN$; [by 24. def. 11.] it will be as BD to FH , so is the axis KL to the axis MN . But as BD is to FH , so is BK to FM : Wherefore as BK is to FM , so is KL to MN . Therefore alternately as BK is to KL , so is FM to MN : And the angles BKL, FMN are equal, as being right angles, and the sides about the equal angles BKL, FMN are proportional. Therefore [by 6. 6.] the triangle BKL is similar to the triangle FMN . Again, because as BK is to KT , so is FM to MO , and they are about the equal angles BKT, FMO ; for the angle BKT is the same part of four right angles which are at the centre K , as the angle FMO is of the four right angles about the centre M : Therefore because the sides about the equal angles are proportional; the triangle BKT [by 6. 6.] will be similar to the triangle FMO . Again, because it has been proved that as BK is to KL , so is FM to MN ; but BK is equal to KT , and FM to MO ; it will be as KT to KL , so is OM to MN , and the sides about the equal right angles TKL, OMN are proportional: Therefore the triangle AKT is similar to the triangle NMO : But because from the similarity of the triangles BKL, FMN , it is as AB is to BK , so is NF to FM , and since the triangles BKT, FMO are similar, as KB is to BT , so is MF to MO ; therefore by equality of ratio [by 22. 1.] as AB is to BT , so is NF to FO . Again, because the triangles LTK, NOM are similar, it is as LT is to TK , so is NO to OM ; and because of the similar triangles KBT, OMF ,

as

as KT is to TE , so is MO to OF ; it will be, by equality of ratio, as LT is to TE , so is NO to OF . But it has been proved that as TE is to EL , so is OF to FN : Wherefore again, by equality of ratio, as TL is to LE , so is ON to NF : Therefore the sides of the triangles ATE , NOF are proportional; and so [by 5. 6.] the triangles ATE , NOF are equiangular and similar to one another: Wherefore [by 9. def. 11.] the pyramid, whose base is the triangle BEK , and vertex the point L , is similar to the pyramid whose base is the triangle FMO , and vertex the point N : For they are contained under equal numbers of similar planes: But similar pyramids that have triangular bases, are [by 8. 12.] in the triplicate ratio of their homologous sides: Therefore the pyramid $BEKTL$ to the pyramid $FMON$ is in the triplicate ratio of BK to FM . In like manner drawing right lines from the points A, Z, D, Q, C, V , to K ; and from E, S, H, R, G, P , to M , and erecting pyramids upon the triangles of the same vertexes as the cones; we demonstrate that each of the pyramids of one cone is to every one of the other, in the triplicate ratio of the homologous side BK to the homologous side FM : that is, of BD to FH . But [by 12. 5.] as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents: Therefore as the pyramid $BEKTL$ is to the pyramid $FMON$, so is the whole pyramid, whose base is the polygon $ATEBVCQDZ$, and vertex the point L , to the whole pyramid, whose base is the polygon $EOFPGRHS$, and vertex the point N : Wherefore the pyramid whose base is the polygon $ATEBVCQDZ$, and vertex the point L , is to the pyramid whose base is the polygon $EOFPGRHS$, and vertex the point N , in the triplicate ratio of BD to FH . But the cone whose base is the circle $ABCD$, and vertex the point L , is supposed to be to the solid x in the triplicate ratio of BD to FH : Therefore as the cone whose base is the circle $ABCD$, and vertex the point L , is to the solid x , so is the pyramid whose base is the polygon $ATEBVCQDZ$, and vertex the point L , to the pyramid whose base is the polygon $EOFPGRHS$, and vertex the point N . And alternately [by 16. 5.] as the cone whose base is the circle $ABCD$, and vertex the point L , is to the pyramid

in

in it, whose base is the polygon $A T E V C Q D$, and vertex the point L , so is the solid x to the pyramid whose base is the polygon $E O F P G R H S$, and vertex the point N . But the said cone is greater than the pyramid in it; for it encompasses it: Therefore the solid x is greater than the pyramid whose base is the polygon $E O F P G R H S$ and vertex N . But it is also less; which is impossible: Therefore the cone whose base is the circle $A B C D$, and vertex the point L , is not to some solid less than the cone, whose base is the circle $E F G H$, and vertex the point N , in the triplicate ratio of $B D$ to $F H$. We demonstrate also, after the same manner, that the cone $E F G H N$ is not to some solid less than the cone $A B C D L$ in the triplicate ratio of $F H$ to $B D$. I say too that the cone $A B C D H$ to some solid greater than the cone $E F G H N$, is not in the triplicate ratio of $B D$ to $F H$. For if possible, let it be so to some solid greater, as x . Then inversely, the solid x is to the cone $A B C D L$ in the triplicate ratio of $F H$ to $B D$. But as the solid x is to the cone $A B C D L$, so is the cone $E F G H N$, to some solid less than the cone $A B C D L$: Therefore the cone $E F G H N$ is to some solid less than the cone $A B C D L$, in the triplicate ratio of $F H$ to $B D$; which has been demonstrated to be impossible: Therefore the cone $A B C D L$ is not to some solid greater than the cone $E F G H N$, in the triplicate ratio of $B D$ to $F H$. But it has been also proved not to be so to a solid less: Therefore the cone $A B C D L$ is to the cone $E F G H N$ in the triplicate ratio of $B D$ to $F H$. But as one cone is to the other, so is one cylinder to the other; for a cylinder having the same base and altitude as a cone, is thrice the cone, since it is proved [by 10. 12.] that every cone is one third part of a cylinder of the same base and altitude. Therefore one cylinder is to the other in the triplicate ratio of $B D$ to $F H$.

Wherefore similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases. Which was to be demonstrated.

^f The demonstration of this theorem immediately follows from that of the eighth, by supposing cones to be pyramids having polygonous bases of infinite numbers of small equal sides, and cylinders to be prisms having such polygonous bases.

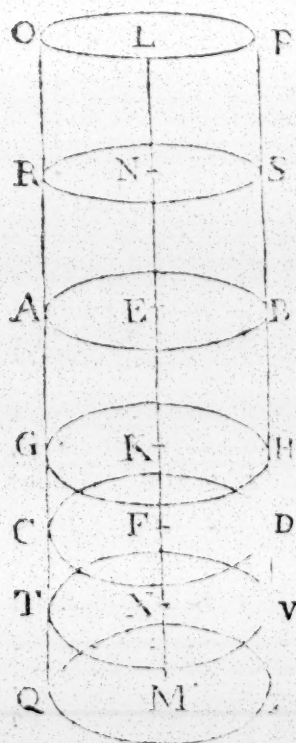
P R O P.

PROP. XIII. THEOR.

If a cylinder be cut [into two cylinders] by a plane parallel to the opposite planes; it will be as one cylinder is to the other, so is the axis of the one to the axis of the other.

For let the cylinder AD be cut by the plane GH [into two cylinders] parallel to the opposite planes AB , CD , meeting the axis EF in the point K : I say as the cylinder BG is to the cylinder GD , so is the axis EK to the axis KF .

For continue out the axis EF both ways to the points L , M , and take any number of right lines EN , NL , each equal to the axis EK , and any number FX , XM equal to KF : And thro' the points L , N , X , M , draw planes parallel to the planes AB , CD , and let the circles OP , RS , TV , QZ , be conceived to be described in the planes drawn thro' L , N , X , M , about the centres L , N , X , M , equal to AB , CD ; and conceive the cylinders PR , RB , DT , TZ to be constituted. Then because the axes LN , NE , EK are equal to one another;



the cylinders PR , RB , BG [by 11. 12.] will be to one another as their bases: But the bases are equal; and therefore the cylinders PR , RB , BG are equal to one another. And because the axes LN , NE , EK are equal to one another, and also the cylinders PR , RB , BG equal to one another; and the multitude of LN , NE , EK , is equal to the multitude of PR , RB , BG ; the axis KL is the same multiple of the axis EK , as the cylinder PG is of the cylinder GB . By the same reason the axis MK is the same multiple of the axis KE , as the cylinder zG is of the cylinder GD . And if the axis KL be equal to the axis KM , the cylinder PG will be equal to the cylinder GZ . If the axis KL be greater than the

axis

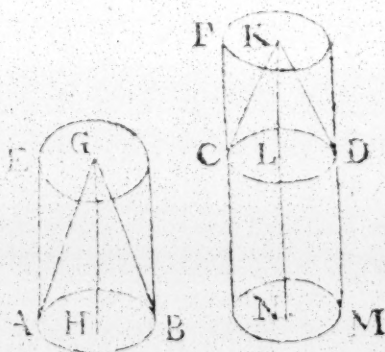
axis KM , the cylinder PG will be greater than the cylinder GZ ; and if less, less. Therefore there are four magnitudes, *viz.* the axes EK , KF , and the cylinders BG , GD , and there are taken equimultiples of the axis EK , and cylinder BG , *viz.* the axis KL , and the cylinder PG ; and equimultiples of the axis KF , and cylinder GD , *viz.* the axis KM and the cylinder GZ . And it has been also demonstrated, that if the axis KL exceeds the axis KM , the cylinder PG will exceed the cylinder GZ ; if it be equal, equal; and if less, less: Therefore [by 5. def. 5.] as the axis EK is to the axis KF , so is the cylinder BG to the cylinder GD . Which was to be demonstrated.

PROP. XIV. THEOR.

Cones and cylinders standing upon equal bases, are to one another as their altitudes.

For let the cylinders FD , EB stand upon the equal bases AB , CD : I say as the cylinder EB is to the cylinder FD , so is the axis GH to the axis KL .

For produce the axis KL to the point N ; and make LN equal to the axis GH ; and about the axis LN let us conceive the cylinder CM to stand. Then because the cylinders EB , CM have the same altitude; [by 11. 12.] they are to one another as their bases: But the bases are equal: Therefore the cylinders EB , CM will be equal. But because the cylinder FM is cut by the plane CD parallel to the opposite planes; the cylinder CM [by 13. 12.] will be to the cylinder FD , as the axis LN is to the axis KL . But the cylinder CM is equal to the cylinder EB , and the axis LN to the axis GH : Therefore as the cylinder EB is to the cylinder FD , so is the axis GH to the axis KL . But as the cylinder EB is to the cylinder FD , so is the cone ABG to the cone CDK ; for [by 10. 12.] they are
thrice

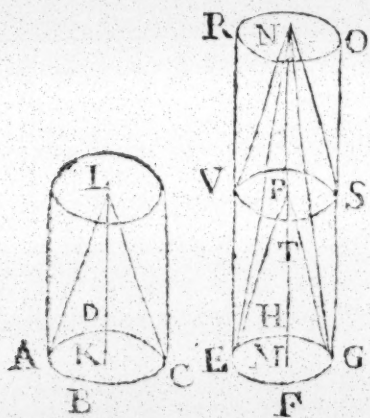


thrice the cones: Therefore as the axis GH is to the axis KL , so is the cone ABG to the cone CDK ; and the cylinder EB to the cylinder FD . Which was to be demonstrated.

PROP. XV. THEOR.

The bases of equal cones or cylinders are reciprocally proportional to their altitudes: also those cones and cylinders whose bases are reciprocally proportional to their altitudes, are equal.

Let there be equal cones or cylinders, whose bases are the circles $ABCD$, $EFGH$, and let AC , EG be their diameters, and KL , MN their axes, which are the altitudes of the cones or cylinders; and compleat the cylinders AX , EO : I say the bases are reciprocally proportional to the altitudes; that is, as the base $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude KL .



For the altitude KL is either equal to the altitude MN , or not equal to it. First let it be equal, and then the cylinder AX is equal to the cylinder EO . But cones and cylinders [by 11.12.] having the same altitude, are to each other as their bases: Therefore the base $ABCD$ is equal to the base $EFGH$, and accordingly they are reciprocally

proportional, that is, as the base $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude KL . Now let the altitude KL not be equal to the altitude MN , but let MN be greater. And from MN take the altitude PM equal to the altitude KL , and cut the cylinder EO thro' P by the plane TVS , parallel to the opposite planes $EFGH$, RO of the circles, and conceive the cylinder ES to be made, whose base is the circle $EFGH$, and altitude PM . Then because [by supposition] the cylinder AX is equal to the cylinder EO ,
and

and ES is some other cylinder; it will be [by 7. 5.] as the cylinder AX to the cylinder ES , so is the cylinder EO to the cylinder ES . But [by 11. 12.] as the cylinder AX is to the cylinder ES , so is the base $ABCD$ to the base $EFGH$. For the cylinders AX , ES have the same altitude. But as the cylinder EO is to the cylinder ES , so [by 13. 12.] is the altitude MN to the altitude MP ; for the cylinder EO is cut by the plane TVS parallel to the opposite planes: Therefore as the base $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude MP . But the altitude MP is equal to the altitude KL : Wherefore as the base $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude KL . Therefore the bases of equal cylinders AX , EO are reciprocally proportional to their altitudes.

But now let the bases of the cylinders AX , EO be reciprocally proportional to their altitudes, and let the base $ABCD$ be to the base $EFGH$, as the altitude MN is to the altitude KL : I say the cylinder AX is equal to the cylinder EO .

For the same construction remaining, because as the base $ABCD$ is to the base $EFGH$, so is the altitude MN to the altitude KL , and the altitude KL is equal to the altitude MP : The base $ABCD$ will be to the base $EFGH$, as the altitude MN is to the altitude MP . But [by 11. 12.] as the base $ABCD$ is to the base $EFGH$, so is the cylinder AX to the cylinder ES ; for they have the same altitude. But as the altitude MN is to the altitude MP , so [by 13. 12.] is the cylinder EO to the cylinder ES : And therefore as the cylinder AX is to the cylinder ES , so is the cylinder EO to the cylinder ES : Therefore [by 9. 5.] the cylinder AX is equal to the cylinder EO , and the like in cones. Which was to be demonstrated.

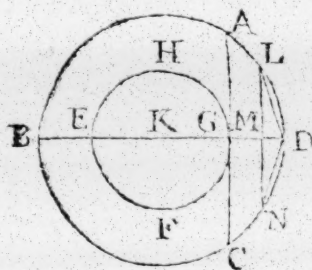
PROP. XVI. PROBL.

Two circles about the same centre being given; to describe a polygon in the greater of them, of an equal and even number sides, which does not touch the lesser circle.

Let $ABCD$, $EFGH$ be two given circles, having the same

same centre K : It is required to describe a polygon of an equal and even number of sides in the greater circle $A B C D$, not touching the lesser circle $E F G H$.

For draw the right line $B D$ thro' the centre K , and from the point G draw $A G$ at right angles to $B D$, which continue out to C : Then [by 16. 3] $A C$ touches the circle $E F G H$; and so bisecting the semi-circumference $B A D$, and again bisecting its half, and doing this continually; [by 1. 10.] there will at last remain some part of the semi-circumference less than $A D$. Let this remainder



be $L D$. And from the point L draw $L M$ perpendicular to $B D$, which continue out to N ; and join $L D$, $D N$. Therefore $L D$ is equal to $D N$. And because $L N$ is parallel to $A C$, and $A C$ touches the circle $E F G H$; $L N$ will not touch the circle $E F G H$, and much less will the

right lines $L D$, $D N$ touch the circle $E F G H$. Therefore if [by 1. 4.] right lines be continually apply'd round the circumference of the circle, each equal to $L D$, there will be described in the circle a polygon of an equal and even number of sides, not touching the lesser circle $E F G H$. Which was to be demonstrated.

PROP. XVII. PROBL.

Two spheres, having the same centre, being given; to describe a solid polyhedron in the greater, not touching the superficies of the lesser sphere.

Conceive two spheres having the same centre A : It is required to describe a solid polyhedron in the greater sphere, not touching the superficies of the lesser sphere.

Let the sphere be cut by some plane passing thro' the centre: Then will the sections be circles: Because [by 14. def. 11.] the sphere is made by the revolution of a circle about its diameter while the diameter remains at rest; so that in whatsoever situation the semicircle is conceived to be, the plane of it being continued, will make a circle in the superficies of the sphere: And it is manifest that the circle is a great circle, because the diameter of the sphere (and that of the semicircle) [by 15. 3.] is greater than all right lines

meeting the superficies of the sphere in the point x ; and draw a plane thro' Ax and Bd , and another thro' Ax and Kx , which, from what has been said, make great circles in the superficies of the sphere. Let such be made, and upon their diameters Bd , Kx let Bxd , Kxn be their semicircles. Then because the right line Ax is perpendicular to the plane of the circle $Bcde$, all the planes which pass by Ax will be [by 18. 11.] perpendicular to the plane of the circle $Bcde$: And so the semicircles Bxd , Kxn are perpendicular to the same plane. And because the semicircles Bxd , Kxn are equal, for they are upon the equal diameters Bd , Kx ; the quadrants Be , Bx , Kx are equal: Therefore as many sides of a polygon as are in the quadrant Be , so many sides there will be in the quadrants Bx , Kx equal to Bk , kl , lm , me . Describe them, and let them be Bo , Op , Pr , Rx , Ks , St , Tv , Vx ; and join so , tp , vr ; and from the points o , s , draw perpendiculars to the plane of the circle $Bcde$: these [by 38. 11.] will fall in the common sections Bd , Kx of the planes; because the planes of the semicircles Bxd , Kxn [by constr.] are perpendicular to the plane of the circle $Bcde$. Let these perpendiculars be oq , sz , and join qz . Then because in the equal semicircles Bxd , Kxn there are taken the equal parts of the circumference Bo , Ks , and there are drawn the perpendiculars oq , sz ; oq will be equal to sz , and Bq equal to Kz . But the whole BA is equal to the whole KA : Wherefore the remainder QA is equal to the remainder ZA : Therefore as BQ is to QA , so is KZ to ZA ; and so [by 2. 6.] QZ is parallel to KB : And because oq , sz are each perpendicular to the plane of the circle $Bcde$; [by 6. 11.] oq will be parallel to sz . But it has been also proved to be equal to it. Wherefore [by 33. 1.] zq , so are equal and parallel. And because zq is parallel to so , and also parallel to KB ; [by 9. 11.] so will be parallel to KB ; and Bo , Ks join them: Therefore the quadrilateral figure $KBoS$ is in one plane; for [by 7. 11.] if two right lines be parallel, and any points be taken in each of them, the right line joining those points is in the same plane as the parallels. And by the same reason the quadrilateral figures $sopt$, $tpRV$ are each in one plane. But [by 2. 11.] the triangle VRx is in one plane: If therefore from the points

points O, S, P, T, R, V right lines be conceived to be drawn to A , there will be made solid polyhedrons composed of pyramids, between the circumferences EX, KX whose bases are the quadrilateral figures $KBO S, S O P T, T P R V$, and the triangle VRX , and vertex the point A . Now if upon each of the sides KL, LM, ME , we make the same construction as on KB , as also in the rest of the three quadrants, and in the remaining hemisphere; there will be described a solid polyhedron in the sphere composed of pyramids, whose bases are quadrilateral figures of the same order with those and the triangle already mentioned, and vertex the point A : I say the said polyhedron does not touch the superficies of the lesser sphere, in which the circle FGH is. Draw [by II. 11.] AY from the point A perpendicular to the plane of the quadrilateral figure $KBO S$, meeting it in the point Y , and join BY, YK . Then because AY is perpendicular to the plane of the quadrilateral figure $KBO S$, [by 3. def. 11.] it will be perpendicular to all right lines that touch it, and are in the same plane: Therefore AY is at right angles to BY , and to YK . And because AB is equal to AK ; the square of AB will be equal to the square of AK . But [by 47. 1.] the squares of AY, YB are equal to the square of AB , for [by constr.] the angle at Y is a right angle; and the squares of AY, YK are equal to the square of AK : Therefore the squares of AY, YB are equal to the squares of AY, YK : Take away the common square of AY , and the square of BY remaining is equal to the square of YK remaining; and so the right line BY is equal to the right line YK . After the same manner we demonstrate that each of the right lines drawn from the point Y to O, S , is equal to BY , or YK . Therefore a circle described about the centre Y with either of the distances YB , or YK , will pass thro' the points O, S ; and so $KBO S$ will be a quadrilateral figure in a circle. And because KB is greater than ZQ , and ZQ is equal to SO ; KB will be greater than SO . But KB is equal to KS or BO : Therefore KS or BO are each greater than SO . Therefore since $KBO S$ is a quadrilateral figure in a circle, and KB, BO, KS are equal, and OS lesser, than either of them, and BY is a line drawn from the centre; the square of KB will be greater than twice the square of BY . Draw the perpendicular KW from the point K to BD .

Then because BD is less than twice DW , and [by I. 6.] as DB is to DW , so is the rectangle contained under DB , BW to the rectangle contained under DW , WB ; wherefore the square of BW being described, and the parallelogram upon WD being compleated, the rectangle under DB , BW is less than twice the rectangle under DW , WB . And moreover joining KD , the rectangle under DB , BW [by 8. 6.] is equal to the square of KB , and the rectangle under DW , WB equal to the square of KW . Therefore the square of KB is less than twice the square of KW . But the square of KB is greater than twice the square of BY : Therefore the square of KW is greater than the square of BY . And because BA is equal to KA , the square of BA will be equal to the square of KA . And [by 47. 1.] the squares of BY , YA are equal to the square of BA , and the squares of KW , WA equal to the square of KA : Therefore the squares of BY , YA are equal to the squares of KW , WA ; but the square of KW is greater than the square of BY : Therefore the remaining square of WA is less than the square of YA ; and accordingly the right line AY is greater than the right line AW : Therefore AY is much greater than AG . But AY is at one base of the polyhedron, and AG at the superficies of the lesser sphere: Wherefore the polyhedron does not touch the superficies of the lesser sphere.

Otherwise.

It may be demonstrated more easily and expeditiously that AY is greater than AG . From the point G draw GL at right angles to AG , and join AL . Then the circumference EB being bisected, and one half of it also bisected, and this being continually done, there will at last [by I. 10.] be left a part of the circumference less than the arch of the circumference of the circle BCD , whose * substance is equal to GL . Let the part of the circumference KB be this ultimate remainder. Then the right line KB is less than GL . And because $BKSO$ is a quadrilateral figure in a circle, and OB , BK , KS are equal, and OS less than either; the angle BYK will be obtuse; and so BK is greater than BY . But GL [by constr.] is greater than BK : Wherefore GL is much greater than BY , and the square of GL is greater than the square of BY . And because AL is equal to AB , the square of AL

* A right line drawn from one end of it to the other.

will

will be equal to the square of AB . But the squares of AG , GL are equal to the square of AL , and the squares of BY , YA equal to the square of AB : Therefore the squares of AG , GL are equal to the squares of BY , YA : But the square of BY is less than the square of GL : Wherefore the remaining square of YA is greater than the remaining square of AG ; and so the right line AY is greater than the right line AG .

Therefore two spheres having the same centre, being given, a solid polyhedron is described in the greater sphere that does not touch the superficies of the lesser sphere. Which was to be demonstrated.

Corollary. Also if a solid polyhedron be inscribed in the lesser sphere similar to that described in the greater sphere $BCDE$: the solid polyhedron in the lesser sphere will be to the solid polyhedron in the greater sphere $BCDE$ in the triplicate ratio of the diameter of the greater sphere $BCDE$ to the diameter of the lesser sphere. For the solids being divided into equal numbers of pyramids of the same order, these pyramids will be similar. And similar pyramids [by cor. 8. 12.] are to one another in the triplicate ratio of their homologous sides: Therefore the pyramid whose base is the quadrilateral figure $KBOs$, and vertex the point A , is to the pyramid in the other sphere of the same order in the triplicate ratio of the homologous side of the one to the homologous side of the other; that is, of the semidiameter AB of one of the spheres, to the semidiameter of the other. In like manner each of the pyramids in one of the spheres to the correspondent pyramid in the other, will be in the triplicate ratio of the semidiameter of one of the spheres to the semidiameter of the other. But [by 12. 5.] as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents: Wherefore the whole solid polyhedron which is in one of the spheres about the centre A , is to the other, in the triplicate ratio of the semidiameter of one of the spheres, to the semidiameter of the other; that is, of the diameter BD of the one, to the diameter of the other.

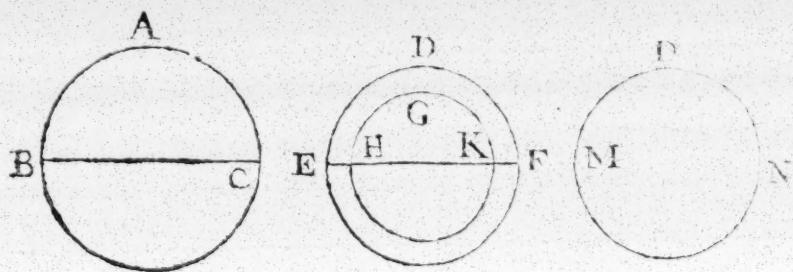
§ This problem, being a sort of unlimited one, is generally stumbled at by young beginners, who seldom or never understand it till they have read the demonstration. It is of no other use in these Elements but to demonstrate the next theorem; and, to tell the truth, is not very inviting in itself.

PROP. XVIII. THEOR.

Spheres are to one another in the tripliate ratio of their diameters^h.

Conceive ABC , DEF to be two spheres, whose diameters are BC , EF : I say the sphere ABC is to the sphere DEF in the tripliate ratio of BC to EF .

For if not, the sphere ABC to some sphere either greater or less than DEF , will be in the tripliate ratio of BC to EF . First let it be so to a sphere which is less, viz. to GHK ; and conceive the sphere DEF to have the same centre as the sphere GHK ; and [by 17. 12.] in the



greater of these spheres DEF describe a solid polyhedron that does not touch the superficies of the lesser sphere GHK ; and in the sphere ABC describe a solid polyhedron similar to that described in DEF : Then the solid polyhedron in the sphere ABC is to the solid polyhedron in the sphere DEF , [by cor. 17. 12.] in the tripliate ratio of BC to EF . But [by supposition] the sphere ABC is to the sphere GHK in the tripliate ratio of BC to EF : Therefore [by 11. 5.] as the sphere ABC is to the sphere GHK , so is the solid polyhedron in the sphere ABC to the solid polyhedron in the sphere DEF . And alternately as the sphere ABC is to the polyhedron which is in it, so is the sphere GHK to the polyhedron in the sphere DEF . But the sphere ABC is greater than the solid polyhedron which is in it; and therefore the sphere GHK is greater than the polyhedron in the sphere DEF : But it is less too, for it is contained in it; which is impossible: Therefore the sphere ABC is not to a sphere less than DEF in the tripliate ratio of BC to EF . In like manner we demonstrate that the sphere DEF is not to some sphere less than ABC in the tripliate ratio of EF to BC . I say also that

that the sphere ABC to some sphere greater than DEF , is not in the triplicate ratio of BC to EF . For if possible, let it be so to some greater sphere LMN : Then inversely the sphere LMN is to the sphere ABC in the triplicate ratio of the diameter EF to the diameter BC . And as the sphere LMN is to the sphere ABC , so is the sphere DEF , to some sphere less than ABC , as has been already demonstrated; because the sphere LMN is greater than DEF ; therefore the sphere DEF is to a sphere less than ABC in the triplicate ratio of EF to BC ; which has been already proved to be impossible. Wherefore the sphere ABC is not to some sphere greater than DEF in the triplicate ratio of BC to EF ; and it has been demonstrated not to be so to some sphere less: Therefore the sphere ABC will be to the sphere DEF in the triplicate ratio of BC to EF . Which was to be demonstrated.

^h This theorem may be demonstrated shorter, by supposing spheres to be similar solids having infinite equal numbers of small equal square faces, the four angles of each square being in the superficies of the spheres; for then a sphere will consist of an infinite number of small equal pyramids, whose vertexes will all be at the centre of the sphere, and small square bases at the superficies of the sphere. So also may any other sphere consist of the like number of small square pyramids, and each of these pyramids in each of the spheres, will be similar; and so will be to one another in the triplicate ratio of the correspondent sides of the pyramid; that is, of the semidiameters of the spheres; and so will all such pyramids in one sphere, be to all in the other; that is, one sphere to the other.

Additions to the Twelfth Book.

PROP. I. THEOR.

If a right line AB be divided into two parts AP , PB . I say the cube of the whole line AB will be equal to the aggregate of the cubes of the parts AP , PB together with three parallelepipeds, each of whose base is the square of one of the parts AP , and altitude the part PB , and three parallelepipeds, each of whose base is the square of the other part PB , and altitude the part AP . That is, $\overline{AB}^3 = \overline{AP}^3 + \overline{PB}^3 + 3\overline{AP}^2 \times PB + 3\overline{PB}^2 \times AP$.

For let us call AB , a ; and AP , x , and PB , z . Then [by 4. 2.] $xx + 2xz + zz = aa$; and drawing each side into x , it will be $x^3 + 2xxx + xzz = xaa$. And drawing each side into z , it will be $xxz + 2xzz + z^3 = zaa$; and putting these two expressions together; it will be $x^3 + z^3 + 2xxx + xzz + 2xzz + xxz = xaa + zaa$. That is, $x^3 + z^3 + 3xxx + 3zzx = xaa + zaa = x + z \times aa = a^3$; since $a = x + z$.

PROP. II. THEOR.

The cube of the difference BP of two right lines AP , AB , is equal to the difference of the aggregate of the cube of the greater line AP , and three parallelepipeds each of whose base is the square of the lesser part AB and altitude the greater part AP , and the cube of the lesser part AB together with the three parallelepipeds whose base is the square of the greater part AP , and altitude the lesser part AB . That is $\overline{AP - BA}^3 = \overline{AP}^3 + 3\overline{AB}^2 \times AP - \overline{AB}^3 - 3\overline{AP}^2 \times AB$.

For

For calling AP, x ; and AB, z ; and BP, d : It will be [by 7. 2.] $xx - 2xz + zz = dd$. And drawing each side into x , we have $x^3 - 2xxz + xzz = dd x$. Also drawing each side into z , and we have $xxz - 2xzz + z^3 = dd z$, and taking the latter of these expressions from the former, we shall have $x^3 - 3xxz + 3xzz - z^3 = xdd - zdd = dd x - z = d^3$: That is $\overline{AP^3} + 3\overline{AB^2} \times AP - 3\overline{AP^2} \times AB - \overline{AB^3} = \overline{AP - AB^3} = \overline{BP^3}$.

PROP. III. THEOR.

If a right line AB be divided into two parts AP, PB . I say the cube of the whole line AB will be equal to the aggregate of the cube of the parts AP, PB , together with three times the parallelepipedon under each part AP, PB , and the whole line AB . That is, $\overline{AP^3} + \overline{PB^3} + 3AP \times PB \times AB = \overline{AB^3}$.

Let us call AP, x ; PB, z ; and AB, a : Then [by prop. 1.] $a^3 = x^3 + z^3 + 3xxz + 3zzx$. That is, since $3xxz + 3zzx = 3x + 3z \times xz = 3a \times xz$. For $a = x + z$; it will be $x^3 + z^3 + 3axz = a^3$. Or $\overline{AP^3} + \overline{PB^3} + 3AB \times AP \times PB = \overline{AB^3}$.

PROP. IV. THEOR.

If a right line AP be divided into two parts AB, BP . I say the cube of the part PB , together with three parallelepipedons under the parts AB, BP , and the whole line AP , will be equal to the difference of the cubes of the whole line AP , and the greater part AB . That is $\overline{BP^3} + 3AB \times BP \times AP = \overline{AP^3} - \overline{AB^3}$.

For call AP, x ; AB, a ; and BP, z . Then will [by prop. 3.] $x^3 = a^3 + z^3 + 3axz$; and taking from each side a^3 , we shall have $x^3 - a^3 = z^3 + 3axz$. That is, $\overline{AP^3} - \overline{AB^3} = \overline{BP^3} + 3AB \times BP \times AP$.

PROP.

PROP. V. THEOR.

If a right line AB be divided into two parts AP, PB . I say the cube of the greater part AP together with the parallelepipedon under the parts AP, PB , and the difference $AP - PB$ of the parts, is equal to the cube of the lesser part PB together with the parallelepipedon whose base is the whole line AB , and altitude the difference of the parts AP, PB .

$\begin{array}{ccc} A & P & B \\ \hline & | & \end{array}$ That is, $\overline{AP^3} + AP \times PB \times \overline{AP - PB} = \overline{PB^3} + \overline{AP - PB} \times \overline{AB^3}$.

For let us call AP, x ; PB, z ; AB, a ; and $AP - PB, b$: Then will $xx + za = zz + xa$. And drawing each side into x ; it will be $x^3 + xza = xzz + xxa$. And drawing each side into z ; it will be $xxz + zza = z^3 + xaz$. And putting together each of these expressions, we shall have $x^3 + xza + xxz + zza = xzz + xxa + xza + z^3$. And taking away xza from both sides, it will be $x^3 + xxz + zza = xzz + xxa + z^3$. And taking zza from both; it will be $x^3 + xxz = xzz + z^3 + xxa - zza$; and again taking away xzz ; it will be $x^3 + xxz - xzz = z^3 + xxa - zza$; that is, $x^3 + x - z \times xz = z^3 + xx - zz \times a$. Or (since $b = x - z$, and $x + z = a$, and $xx - zz = x + z \times x - z$) $x^3 + bxz = z^3 + baa$; or $\overline{AP^3} + AP \times PB \times \overline{AP - PB} = \overline{PB^3} + \overline{AP - PB} \times \overline{AB^3}$.

PROP. VI. THEOR.

If a right line AB be divided into two parts AP, PB . I say the aggregate of the cubes of the parts will be equal to a parallelepipedon, whose base is the difference between the aggregate of the squares of AP, PB , and the rectangle under them, and altitude the whole line AB .

$\begin{array}{ccc} A & P & B \\ \hline & | & \end{array}$ That is, $\overline{AP^3} + \overline{PB^3} = \overline{AP^2 + PB^2 - AP \times PB} \times \overline{AB}$.

Let us call AP, x ; PB, z ; and AB, a : Then [by prop. 3.] $x^3 + z^3 + 3axz = a^3$. And taking away $3axz$ from each side; it will be $x^3 + z^3 = a^3 - 3axz$

$= aa - 3xz \times a$. And because [by 4. 2. Eucl.] $xx + 2xz + zz = aa$. If from each side be taken $3xz$, it will be $xx + 2xz + zz - 3xz$ that is, $xx + xz - xz = aa - 3xz$. And so $xx - xz + zz \times a = a^2 - 3xz \times a = a^3 - 3axz$. But since $x^3 + z^3 + 3axz = a^3$, and $x^3 + z^3 = a^3 - 3axz$. Therefore $x^3 + z^3 = xx + xz - xz \times a$. Or $\overline{AP^3} + \overline{PB^3} = \overline{AP^2} + \overline{PB^2} - AP \times PB \times AB$.

PROP. VII. THEOR.

If a right line AP be divided into two parts AB, PB . I say the difference of the cubes of the parts AB, PB will be equal to the parallelepipedon whose base is the aggregate of the squares of the parts AB, PB , and the rectangle under them, and altitude the difference of the parts AB, PB . That is,

$$\overline{AB^3} - \overline{PB^3} = \overline{AB^2} + \overline{PB^2} + AB \times PB \times AB - PB.$$

Let us call AB, x ; PB, z ; and $AB - BP, a$: Then [by prop. 4] $x^3 - z^3 = a^3 + 3axz = aa + 3xz \times a$. But because [by 7. 2. Eucl.] $xx - 2xz + zz = aa$. If to each side be added $3xz$; it will be $xx + xz + zz = aa + 3xz$; and so $xx + xz + zz \times a = aa + 3xz \times a$. Therefore $x^3 - z^3 = xx + xz + zz \times x - z$ (for $x - z = a$) That is $\overline{AB^3} - \overline{BP^3} = \overline{AB^2} + \overline{BP^2} + AB \times BP \times AB - BP$.

PROP. VIII. THEOR

If a right line AB be divided into two equal parts AC, CE and into two unequal parts AP, PB . I say twice the cube of the half line AC or CB , together with three times the parallelepipedon whose base is the square of the intermediate part PC , and altitude thrice the whole line AB , will be equal to the aggregate of the cubes of the parts AP, PB . That is, $2\overline{AC^3} + 3\overline{AB} \times \overline{AP} \times \overline{PC} = \overline{AP^3} + \overline{PB^3}$.

Let

Let us call $AC (=CE) a$; and PC, z . Then this is no other than putting the cube of the aggregate of two lines $a+z$ to the cube of their difference $a-z$. Now $\overline{a+z}^3 = a^3 + 3aaz + 3zza + z^3$ [by 1. of this.] And $\overline{a-z}^3 = a^3 - 3aaz + 3zza - z^3$ [by 2. of this.] And so $\overline{a+z}^3 + \overline{a-z}^3 = 2a^3 + 6aaz$. That is (since $a+z = PB$, and $a-z = AP$.) $\overline{PB}^3 + \overline{AP}^3 = 2\overline{AC}^3 + 3AD \times \overline{PC}^2$.

PROP. IX. THEOR.

If a right line AD be divided into two equal parts, AC, CD , and to the same be added the right line DB . I say twice the cube of the half line AC or CD , together with three times the parallelepipedon whose base is the square of the added line, and the half line taken together, and altitude the whole line, will be equal to the difference between the cubes of the whole and added line taken together, and the cube of the added line. That is,

$$\begin{array}{c} A \quad C \quad D \qquad \qquad \qquad B \\ \hline \overline{AB}^3 - \overline{DB}^3 \end{array} = 2\overline{AC}^3 + 3AD \times \overline{CB}^2$$

Let us call AC or CD, a ; and CB, z : Then [by 1. of this] $a^3 + 3aaz + 3zza + z^3 = \overline{AB}^3$. And $z^3 - 3aaz + 3zza - a^3 = \overline{DB}^3$. And taking the difference of these two equalities, we shall have $2a^3 + 6aaz = \overline{AB}^3 - \overline{DB}^3$. That is (since $AD = 2a$) $2\overline{AC}^3 + 3AD \times \overline{CB}^2 = \overline{AB}^3 - \overline{DB}^3$.

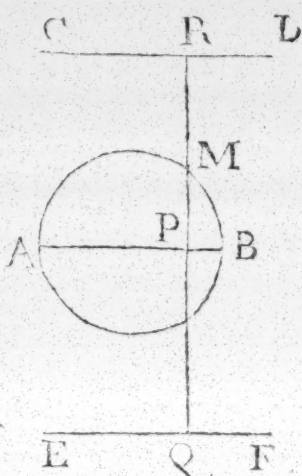
SCHOLIUM.

After the same way, by adding together those two equalities we shall have $2\overline{CB}^3 + 6CB \times \overline{AC}^2 = \overline{AB}^3 + \overline{DB}^3$. There are many other theorems relating to the equalities between the cubes of right lines divided into equal and unequal parts. But these are the most simple, elegant, and useful. --- When I speak of a parallelepipedon, I mean a right angled one; and when I say it is contained under three right lines, I mean the unequal right lines being one side of each face of that solid.

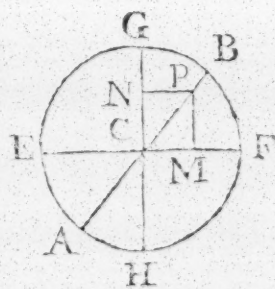
The nine propositions foregoing of the equality of the cubes and parallelepipedons described upon right lines, and the

the segments of them, are very necessary in solid geometry, and some of them new: But their geometrical demonstrations from the solids, after the manner of Euclid, are too long and tedious to be liked by the generality of readers, especially those much addicted to algebra. I have therefore, for brevity's sake, demonstrated them algebraically, believing the geometers will excuse me for so doing. Their use is necessary in the demonstrations of such propositions as these four following, of which I have a great many others in a treatise concerning the equalities and relations of the cubes of right lines drawn in and about a circle.

Prop. 1. If CD, EF be right lines parallel to the diameter AB of a circle, and their distance from it be equal to that diameter. And if any point M be taken in the circumference of the circle, and thro' the same be drawn the right line $RMPQ$ perpendicular to AB , cutting AB in P , and the right lines CD, EF in R and Q . I say four times the cube of the diameter AB will be equal to twice the sum of the cubes of the segments AP and PB together with the sum of the cubes of the parts QM, MR : That is $4\overline{AB}^3 = 2\overline{AP}^3 + 2\overline{PB}^3 + \overline{QM}^3 + \overline{MR}^3$.



Prop. 2. If EF, GH be two diameters of a circle cutting one another at right angles at the centre C , and if any point P be taken within the circle, and two perpendiculars PM, PN be drawn to those diameters; and if a diameter $ACPB$ be drawn thro' P . I say the sum of the cubes of the parts EM, MF, HN, NG , of the diameters EF, HG , will be equal to the sum of the cubes of the parts AP, PB of the diameter AB , together with twice the cube of the semidiameter AC : That is, $\overline{EM}^3 + \overline{MF}^3 + \overline{HN}^3 + \overline{NG}^3 = \overline{AP}^3 + \overline{PB}^3 + 2\overline{AC}^3$.



Prop.

of three angles of a pentagon meeting in one point, *viz.* of a dodecahedron. A solid angle may also be made of three angles of a square meeting in one point, *viz.* of a cube. But besides these there can be no others. For six angles of an equilateral triangle make four right angles; consequently a solid angle cannot consist of six of these plane angles; and much less of more.

A solid angle cannot consist of four right angles, *viz.* those of a square, and much less of more; four angles of a pentagon, are greater than four right angles; and so a solid angle cannot consist of four of these plane angles; and much less of more. Nor can a solid angle be made of the plane angles of any other regular polygons whatsoever; for three angles of a hexagon make four right angles; and so a solid angle cannot be made of them; and much less can it be made of more of them.

But since the three angles of an hexagon are equal to four right angles, the three angles of any other figures of more sides than an hexagon, &c. will be greater than four right angles. And therefore the plane angles of all the regular polygons whose sides are greater in number than five, will be unfit to constitute a solid angle. Wherefore there can be no other regular solids but those five above-mentioned.

Therefore, &c. Which was to be demonstrated.

PROP. XI. PROBL.

The triangular base of a triangular pyramid being given, and each of the three angles that its sides make with the plane of the base: To find the altitude of that pyramid.

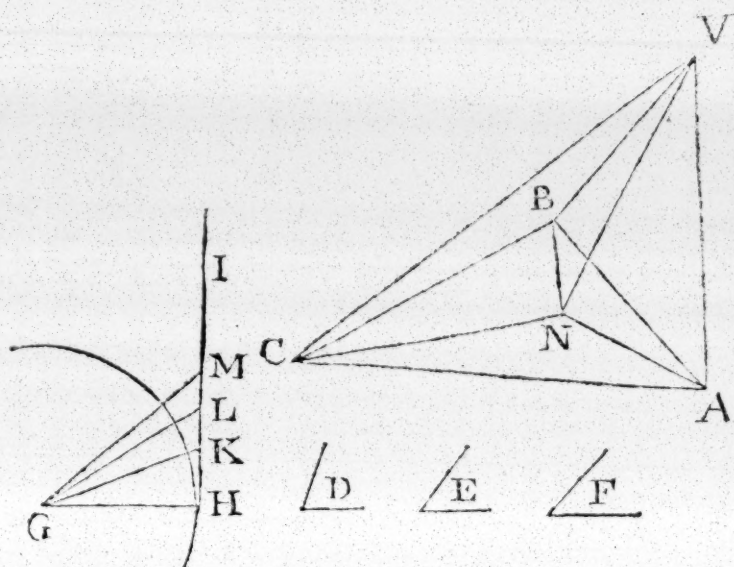
Let ABC be the given triangular base; the angle E equal to the angle at A that one of the sides of the pyramid makes with the base; the angle D equal to the angle at B , that one of the sides of the pyramid makes with the base, and the angle F equal to the angle at C , that one of the sides of the pyramid makes with the base.

With any semidiameter GH describe a circle; and at H raise the perpendicular HI : make the angles HCK , HGL , HGM respectively equal to the excess of a right angle above the given angle D , the given angle E , and

the

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the given angle F ; and draw the right lines GK, GL, GM cutting HI in the points K, L, M . This done, find [by the scholium to prop. 16. of the additions to the sixth book] the point N such, that drawing three right lines BN, AN, CN from the angles of the base to it, they shall be in the respective ratios of HK, HL, HM . Draw NV perpendicular to AN, BN , or CN , and at the point A make the angle NAV equal to the given angle E ; or at



at the angle NBV equal to the given angle D , or at C , the angle NCV equal to the given angle F . Either of these three operations will give the point V , in the perpendicular NV , such, that NV will be the required height of the pyramid. The demonstration almost intirely depends upon the construction, and prop. 4. of the sixth book.

SCHOLIUM.

This problem may be of use in finding the altitude of an object at three stations; it may be solv'd arithmetically. But I leave others to do this.

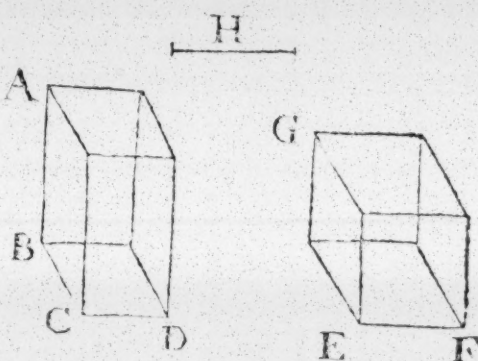
PROP. XII. PROBL.

To make a right angled parallelepipedon of a given base or altitude equal to a given cube.

Let EFG be the given cube, and first let AB be the given altitude which the right angled parallelepipedon to be made equal to the cube is to have.

Now

Now [by 11. 6] find a third proportional BC to the altitude AB and side EF of the cube, and make the rectangle BD under the third proportional BC and the line CD equal to the base of the cube; and upon BD raise the right angled parallelepipedon having the given altitude AB . I say this is equal to the given cube. For because the right angled parallelepipedon ABD is contained under the three right lines AB, CD, BC ; that is, under AB, EF, BC , continual proportionals; the parallelepipedons will [by 36. 11.] be equal to the cube described from the mean line EF .



2dly. Let BD be the given base. And what proportion the given base BD has to the base of the given cube, let the side EF of the cube have to the right line AB [which may be done, if upon the side EF of the cube be made a rectangle equal to the base BD , and upon the other side of that rectangle another rectangle be made equal to the base of the cube: for then it will be [by 1. 6.] as the first rectangle, is to the second rectangle; that is, to the square base of the cube, so is the base of the first rectangle, viz. EF , to the base of the second rectangle]. For if upon the base BD be erected a parallelepipedon having AB for its altitude, the parallelepipedon, and the cube will be [by 34. 11.] equal: because the bases and altitudes [by construction] are reciprocally proportional.

SCHOLIUM.

The converse of this problem, to make a cube equal to a given right angled parallelepipedon cannot be effected from any of the propositions of these elements: that is, it cannot be performed by a right line and a circle: it requires the finding of two mean proportionals between two given right lines. If this could be done, the problem is easily resolved thus: Between BC the side of the square base BD , and the altitude AB of the parallelepipedon, find two mean proportionals EF, H ; so that BC be to EF , as EF is to H , and as H is to AB :

BC

AB

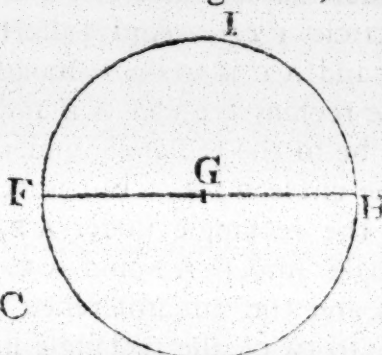
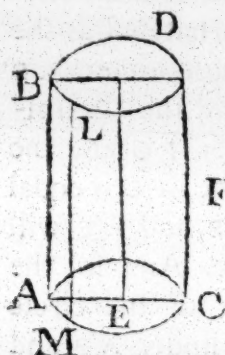
AB: and upon EF construct a cube EFG; which will be equal to the given parallelepipedon ABCD.

So likewise there is no finding the side of a cube that shall be in a given ratio to a given cube till two mean proportionals can be found between two given right lines.

P R O P. XIII.

A circle FIH, the semidiameter FG of which is a mean proportional between the side AB of a right cylinder and the diameter AC of the base, will be equal to the convex superficies of that cylinder.

For first. The convex superficies of a right cylinder is equal to the rectangle under the altitude and the circumference of its base: because if a very small part AM of the circumference of the base be taken for a right line, and perpendiculars AB,



ML to the base be drawn from its extremes to the opposite base: the superficies of the cylinder may be conceived to be made up of an infinite number of such small equal rectan-

gles as ABLM, whose common altitude is the side AB of the cylinder: and all the equal bases put together are equal to the circumference of the base. Therefore the sum of all these rectangles, that is, the convex superficies of the cylinder, will be equal to the rectangle under the side AB of the cylinder and the circumference of the base.

2dly. The circumferences of circles are to one another as their diameters: because if circles be taken for regular similar polygons of equal infinite numbers of sides, and so their circuits proportional to two correspondent sides, and these proportional to the equal lines drawn from the centre to the correspondent angles of the polygon, or to the middles of the sides: the circumferences of circles will be to one another, as the semidiameters or as the diameters.

3dly. A circle is equal to the rectangle under its circumference, and one fourth part of the diameter or under half its circumference, and one half its diameter, or one fourth of

of the rectangle under the circumference and diameter : this follows from a circle being a regular polygon of an infinite number of sides.

These three things being therefore proved, we demonstrate the theorem thus. It will be as the semidiameter AE of the base, is to the circumference AMC , so is the semidiameter FG of the circle FIH , to its circumference. And so by [I. 6.] as the rectangle under AB , AE , is to the rectangle under AB and the circumference of the base, so will the square of FG be to the rectangle under FG , and the circumference of the circle FIH . But since [by construction] the square of FG is equal to the rectangle under AB and AC , or under AB and $2EC$: it will be as the rectangle under AB and the semidiameter of the base, is to the rectangle under AB and the circumference of the base, so will the rectangle under AB and the diameter AC be to the rectangle under the semidiameter FG and the circumference of the circle FIH . And [alternately] as the rectangle under AB and AE is to the rectangle under AB and AC , so will the rectangle under AB and the circumference of the base, be to the rectangle under FG and the circumference of the circle FIH . But since AC is equal to $2AE$, therefore the rectangle under AB and AC will be twice the rectangle under AB and AE . And so the rectangle under FG and the circumference of the circle FIH will be equal to twice the rectangle under AB and the circumference of the base of the cylinder ; or the rectangle under AB and the circumference of the base ACM will be equal to half the rectangle under FG and the circumference of the circle FIH . But the rectangle under AB and the circumference of the base, is equal to the convex superficies of the cylinder, and the rectangle under FG and one half the circumference of the circle FIH is equal to the circle FIH ; therefore the circle FIH will be equal to the convex superficies of the cylinder ABD .

Therefore &c. Which was to be demonstrated.

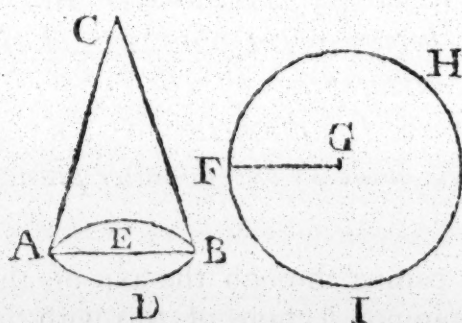
Note, There is no finding a circle equal to the convex superficies of an oblique cylinder, by the Euclidean geometry, viz. by a right line and a circle.

PROP. XIV. THEOR.

A circle FHI the semidiameter FG of which is a mean proportional between the side AC of a right cone ACBD, and the semidiameter AE of the base ABD; will be equal to the convex superficies of that cone.

As in the last proposition the convex superficies of a right cylinder was conceived to consist of an infinite number of equal rectangles all of the same altitude with that of the cylinder, so here the convex superficies of a right cone may be supposed to be made up of an infinite number of equal isosceles triangles, whose bases are infinitely small equal parts of the circumference of the base of the cone, and altitudes the side of the cone. And therefore the convex superficies of a right cone will be equal to half the rectangle under the circumference of the base and side of the cone.

Now, because the circumferences of circles are to one another, as their diameters or semidiameters: [see the demonstration of the last proposition] therefore as the semidiameter AE of the base of the cone, is



to its circumference; so will the semidiameter FG of the circle FHI be to its circumference: and [by I. 6.] as the rectangle under the side AC of the cone, and the semidiameter AE of its base, is to the rectangle

under the side AC and the circumference of the base; so will the square of the semidiameter FG of the circle FHI, be to the rectangle under that semidiameter FG and the circumference of the circle FHI. But because [by construction] the rectangle under AC and AE is equal to the square of FG; it will be as the rectangle under AC and AE, is to the rectangle under AC and the circumference of the base, so will the rectangle under FG and the circumference of the circle FHI. Wherefore the rectangle under AC and the circumference of the base, will be equal to the rectangle under FG and the circumference of the circle FHI. But half the rectangle under AC and the circumference

ference of the base, is equal to the convex superficies, of the right cone $ACBD$. And therefore half the rectangle under FG and the circumference of the circle FHI will be equal to the convex superficies of that cone. But half the rectangle under FG , and the circumference of the circle FHI will be equal to the circle FHI [see the demonstration of the last proposition]. Wherefore the circle FHI , will be equal to the convex superficies of the right cone $ACBD$.

Therefore, &c. Which was to be demonstrated.

SCHOLIUM.

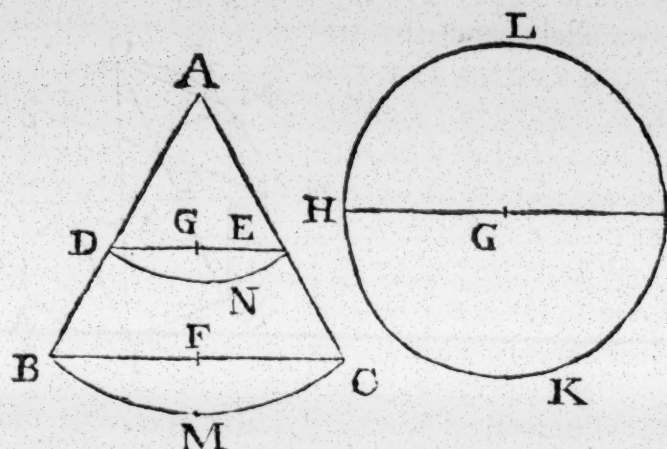
There is no finding a circle equal to the convex superficies of an oblique cone by plain geometry, or even by the help of the conic sections: it cannot be done unless by the invention of a right lined space equal to that of a mixed lin'd space, where the curve partly bounding it is one of the third order.

PROP. XV. THEOR.

The superficies $DBCE$ of a right cone ABC , intercepted by two parallel planes BMC , DNE , will be equal to a circle HIK whose semidiameter HG is a mean proportional between the part BD of the side AB of the cone intercepted by those parallel planes, and the sum of the semidiameters DG , BF of the circles DNE , BMC made by the parallel planes, in the cone.

Let ABC be a triangle passing through the axis of the cone, and BC , DE the common sections of this with the parallel planes. Then [by 16. 11.] BC , DE will be parallel.

Now because [by supposition] BD is to HG , as HG is to the sum of DG and BF ; the square of HG will [by 16. 6.] be equal to the rectangle under BD and the sum of DG and BF ; that is, the square of HG will be equal to the sum of the rectangles under AB and BF , and under AB and DG , and the difference between the sum of the rectangles under AD and DG , and under AD and BF . But because [by 6. 6.] AB is to AD , as BF is to DG , and so [by 16. 6.] the rectangle under AB and DG equal to the rectangle under AD and BF . Therefore the square of HG will be equal to the difference of the rectangles under AB and BF , and under AD , DG . But since [by prop. 2. of this] a circle whose semidiameter is a mean proportional



between $A B$ and $B F$ is equal to the convex superficies of the cone $A B M C$, and a circle whose femidiameter is a mean proportional between $A D$ and $D G$ is equal to the convex superficies of the cone $A D N E$; and since the difference of these circles is equal to the superficies $B C D E$ intercepted by the parallel circles $B M C$, $D N E$; and because [by 2. 12.] circles are to one another as the squares of their femidiameters; therefore the circle $H L K$, whose femidiameter is $H G$, will be equal to the difference of those two circles above-mentioned; that is, equal to the superficies $B C D E$ of the cone, intercepted by the parallel circles $B M C$, $D N E$.

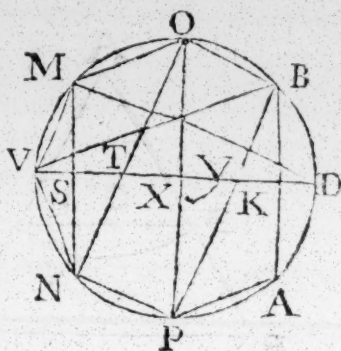
Therefore, &c. Which was to be demonstrated.

P R O P. XVI.

The superficies of the solid $V M O B A P N$ (excluding the base) inscribed in the segment $B V A$ of a sphere and generated by the revolution of the equilateral figure $V M O B A P N$ of an even number of sides inscribed in the segment $B V A$ of a circle, about the axis $V K$, (being part of it,) is equal to a circle whose femidiameter is a mean proportional between the axis $V K$ and the right line $D M$, drawn from the end of the diameter $V D$, to the end of the side $V M$ terminating at the vertex V .

Join the angles on both sides equally distant from the vertex V , by the right lines $M N$, $O P$, $B A$, and join $N O$, $P E$, $D M$. Then will the angles $V D M$, $V M N$, $M N O$, $N O P$ standing upon equal arches [by 27. 3.] be equal to
one

one another. And so [by 27. 1.] the right lines MN , OP , BA will be parallel; and the triangles MDV , MSV , NST , OXT , PXY , BKY , will be similar. Wherefore [by 4. 6.] as DM is to MV ; so will MS be to VS ; NS to ST ; OX to TX ; PX to XV ; BK to YK . Wherefore [by 12. 5.] as DM is to MV , so is the sum of the antecedents MN , OP , BK , to the sum of the respective consequents VK . Therefore [by 16. 6.] the rectangle under DM and VK will be equal to the rectangle under MV and the sum of MN , OP , BK . Now the superficies of the cone MVN is [by 2. of this] equal to a circle, whose semidiameter is a mean proportional between VM and MS ; and the superficies $OMNP$ is [by 3. of this] equal to a circle whose semidiameter is a mean proportional between MO or VM and the sum of MS , OX . Also the superficies $BOPA$ to a circle, whose semidiameter is a mean proportional between OB , or VM , and the sum of OX and BK . Therefore [by 2. 12.] the whole superficies $VMOBAPN$ is equal to a circle, whose semidiameter is a mean proportional between VM and twice MS , twice OX , and twice BK ; that is, a mean proportional between VM and the sum of MN , OP , and twice BA ; that is, a mean proportional between DM and VK .



Therefore, &c. Which was to be demonstrated.

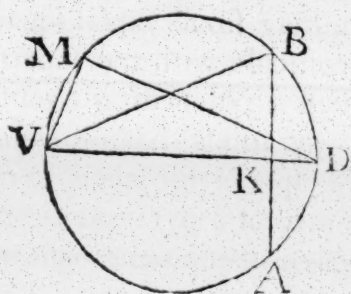
Cor. If VB be drawn. Because the rectangle under DV and VK is equal to the square of VB ; and DM is less than DV ; it is manifest the rectangle under DM and VK will be less than the square of VB ; and so a circle whose semidiameter is a mean proportional between DM and VK will be less than a circle whose semidiameter is VB . That is, the superficies of the solid generated, as above, is less than a circle, whose semidiameter is VB .

P R O B. XVII.

The superficies of a sphere is equal to four times the circle, by the revolution whereof about its diameter, the sphere is generated.

For if a figure of an even number of equal sides, where-

of VM is one, be inscribed in the segment BVA of a circle by the revolution of which about the axis the segment of the sphere is generated; and by the rotation of this figure be produced a solid figure, having its superficies consisting of conical, or partly cylindrical superficies, like as in the last theorem: it is manifest, from the corollary of



that theorem, that the superficies of this figure is less than a circle whose semidiameter is VB . But as the magnitude of VM becomes less, and DM consequently increases, they approach more to an equality; therefore if VM be put infinitely small, the plane figure BVA will coincide with

the segment BVA of the circle, and the solid figure generated, as above with the segment BVA of the sphere, and DM , with the diameter DV ; and accordingly the superficies of the inscribed solid, that is, of the segment BVA of the sphere, will be equal to a circle whose semidiameter is VB ; that is, whose semidiameter is the right line VB drawn from the vertex V of the segment to the circumference of the base BA of the segment. Therefore the superficies of the whole sphere will be equal to a circle, the semidiameter of whose base is the diameter DV of the sphere. For in this case the segment BVA of the sphere is supposed to become the whole sphere; the points A, K, B, D all coinciding. But [by 2. 12.] the circle whose semidiameter is VD , will be four times the circle, whose diameter is DV ; that is, four times the circle, by the revolution of which about its semidiameter the sphere is described.

Which was to be demonstrated.

Hence the superficies of a sphere is equal to the curve superficies of a cylinder circumscribing the sphere, and the superficies of any segment of a sphere, cut off by a plane parallel to the base of such a cylinder, is equal to the correspondent part of the superficies of that cylinder cut off by the plane aforesaid.

P R O P. XVIII.

A sphere will be two third parts of its circumscribing cylinder.

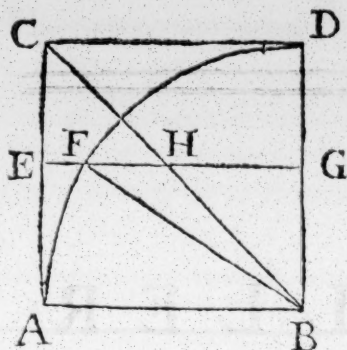
For since every sphere may be conceived to be equal to a polyhedron inscribed in it, or circumscribed about it; having an infinite number of infinitely small equal square faces; and all these square faces coincide with the superficies of the sphere, and because such a polyhedron consists of an infinite number of equal pyramids having equal square bases, and the center of the sphere for their common vertex; also because the solidity of a pyramid is one third part of that of a parallelepipedon of the same base and altitude [by 7. 12.]. And the semidiameter of the sphere may be taken for this altitude. But the solidity of such a polyhedron will be equal to a solid whose base is the sum of all the small square bases abovesaid; and altitude one third part of the semidiameter of the sphere. Therefore the sphere will be equal to a cylindrical solid whose base is the superficies of the sphere, and altitude one third part of the semidiameter of the sphere; or two third parts of the whole diameter. And therefore [by prop. 5.] the sphere will be equal to a cone, the semidiameter of whose base is the diameter of the sphere, and altitude the semidiameter of the sphere. Or the sphere will be equal to a cone whose base is the circle, by the revolution of which about its diameter, the sphere is described, and altitude four times the semidiameter of the sphere, or twice the diameter. Wherefore [by 10. 12.] the sphere will be equal to a cylinder, whose base is the circle, having for its diameter that of the sphere, and altitude one third part of twice the diameter of the sphere, or two third parts of the diameter. But this cylinder is that circumscribing the sphere.

Therefore, *Q. E. D.* Which was to be demonstrated.

Otherwise.

Let $A D B$ be a quarter of a circle; the center being C ; and let $A C D B$ be a square circumscribing it; and $C B$ its diagonal; and let this square and quarter of a circle revolve quite about the fixed axis or semidiameter $A B$; then will the square $A C D B$ describe a cylinder, the quarter $A D B$ of the circle, will describe half a sphere; and the triangle

$D C B$,



DCB, a cone; all having the same altitude AC or DB; and equal circular bases, whose semidiameters are CD, AB.

Now let a plane parallel to the bases cut thro' all these three solids, making circular sections in them; let the semidiameters of these sections be EG, FG, HG.

Join FB; then because of the right angle at G, the square of FB will be equal to the two squares FG, GB; that is, since FB is equal to AB, and EG equal to AB, and because HG is equal to GB; therefore the square of EG will be equal to the squares of FG and GH. As this will always be so, wherever the plane parallel to the base AB or CD cuts the cone, sphere, and cylinder; therefore [by 2. 12.] the circular section of the cylinder, whose semidiameter is EG or AB, will be always equal to both the correspondent circles in the sphere and cone, whose semidiameters are FG, HG. Consequently since this is always so, and because the three solids have the same altitude, the cylinder will be equal to the half sphere, and to the cone, generated as above. But the cone is one third part of that cylinder; therefore the remaining half sphere will be two third parts of that cylinder; and so twice that half sphere; that is, the whole one, will be equal to two third parts of twice that cylinder; that is, the circumscribing cylinder.

Therefore, &c. Which was to be demonstrated.

Cor. Hence the segment of a sphere, whose altitude is DG, and semidiameter of the base is FG, is equal to the difference between a cylinder, the semidiameter of whose base is EG or AB, and the frustum of a cone of the same altitude, the semidiameter of the greatest base of the frustum being CD or AB, and that of the lesser HG.

THE END.

T H E
B O O K S E L L E R
T O T H E
R E A D E R.

Out of respect to my friend and old acquaintance the author, Mr. STONE, and as an extraordinary instance of the acquisition of knowledge from mere self-application, I have ventured to add here his history, given by the *Chevalier de Ramsay*, author of *The travels of Cyrus*, in a letter sent to Father *Castel*, a Jesuit at *Paris*, and published in the *Mémoires de Trevoux*, p. 109.

LETTRE de M. le Chev. de RAMSAY,
membre de la Société Royale d'Angleterre,
au P. CASTEL, *membre de la même*
Société.

LE véritable genie surmonte tous les désavantage de la fortune, de la naissance, de l'éducation; M. STONE en est un rare exemple: né fils du jardinier du Duc d'*Argyle*, il parvint à l'âge de 8 ans, sans sçavoir lire; son pere ne sçavoir pas lui apprendre

dre son métier de cette manière élevée, qui rend le jardinage et l'agriculture une partie très-utile et très-noble de la cosmographie et de la physique.

Par hazard un domestique ayant appris au jeune STONE les lettres de l'alphabet, il n'en fallut pas davantage pour faire éclore son génie et pour le développer. Il s'appliqua lui-même, il étudia, il parvint aux connoissances de la plus sublime géométrie, et du calcul, sans maître, sans conducteur, sans autre guide que la pur génie.

A l'âge de 18 ans il avoit fait tous ces progrès sans être connu, et sans connoître lui même les prodiges qui se passaient en lui.

Milord Duc d'Argyle, qui joint a toutes les vertus militaires, et a tous les sentimens d'un héros, un connoissance universelle de tout ce qui peut orner et perfectionner l'esprit d'un homme de son rang, se promenant un jour dans son jardin, vit sur l'herbe le fameux livre du Chevalier *Newton* en Latin. Il appella quelqu'un pour le ramasser et le reporter dans sa bibliothèque.

Le jeune jardinier lui dit que ce livre lui appartenait : *A vous ?* répondit le Milord. Etendez vous la géométrie, le Latin, M. *Newton*. *J'entends un peu de tout cela*, repliqua STONE avec un air de simplicité fondé sur l'ignorance profonde de ses propres talens, et de l'excès de son sçavoir.

Milord Duc fut très-surpris : mais comme il a le goût des sciences, il daigna entrer en conversation avec le nouveau géometre : Il lui fit plusieurs questions, et demeura étonné de la force, de la justesse, et de la candeur de ses réponses.

Mais comment, dit le Milord, *es tu parvenu a toutes ces connoissances ?* L'autre répondit : *Un domestique m'apprit, il y a dix ans, a lire : A t-on besoin de sçavoir*
voir

voir autre chose que les 24 lettres pour apprendre tout ce que l'on veut ? La curiosité du Duc redouble ; il soupçonne que les démarches de ce genie merveilleux étoient plus suprenantes que ses progrès ; il s'assieoit sur un banc, et lui demande le détail de tout ce qu'il a fait pour devenir habile.

J'appris d'abord à lire, dit Stone, les maçons travailloient alors à votre maison : Je m'approchai d'eux un jour, et je vis que l'architecte usoit d'un regle, d'un compas, et qu'il calculoit. Je demandai ce qu'il faisoit là, et à quoi tout cela étoit bon ; et je compris qu'il y avoit une science que l'on appelloit Arithmetique ; j'achetai un livre d'arithmetique, et je l'appris.

On m'avoit dit, qu'il y en avoit un autre appelée geometrie : j'achetai des livres, et j'appris la geometrie. Je vis à force de lire, qu'il y avoit de beaux livres sur ces deux sciences en Latin : J'achetai un Dictionnaire, et j'appris le Latin. J'appris aussi qu'il y avoit de beaux livres de même espece en François : J'achetai un Dictionnaire, et j'appris le François. Voilà, Monseigneur, tout ce que j'ai fait, il me semble qu'on peut tout apprendre quand on sçait les vingt quatre lettres de l'alphabet.

Ce recit charma Milord Duc. Il tira ce genie merveilleux de l'obscurité ; et il le pourvut d'un emploi qui lui laisse tout le temps de s'appliquer aux sciences. Il decouvrit en lui le même génie pour la musique, pour la peinture, pour l'architecture, pour toutes les sciences qui dépendent du calcul et des proportions.

J'ai vû le Sieur STONE. C'est un homme d'une simplicité admirable. Il sçait à présent qu'il sçait : mais il n'en est pas enflé. Il est possédé d'un amour pur et désintéressé pour la geometrie. Il ne se soucie pas de passer pour geometrie. Le bel esprit et la vanité, n'ont aucune part aux travaux infinis qu'il subit pour exceller dans ce genre. Il méprise aussi la fortune, et m'a sollicité vingt fois de prier Milord de

lui donner un moindre emploi, qui ne valloit que la moitié de celui qu'il a, afin d'être plus solitaire, et moins distrait de ses études favorites. Il découvre quelquefois, par des méthodes qui lui sont propres le mêmes verites que d'autres ont déjà trouvées. Il est charmé de voir qu'il n'en est pas l'inventeur, et que les hommes ont fait plus de progrès qu'il ne croioit. Loin d'être plagiaire, il attribuë les solutions ingenieuses et admirables, qu'il donne de certains problèmes, aux indices qu'il en trouve dans les autres, quoiqu'elles n'en decoulent que par des conséquences fort éloignée, etc.

